# SUPPLEMENT TO THE PAPER "COVARIANCE MATRIX ESTIMATION AND LINEAR PROCESS BOOTSTRAP FOR MULTIVARIATE TIME SERIES OF POSSIBLY INCREASING DIMENSION"

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ABSTRACT. In this supplementary material to Jentsch and Politis (2014) (subsequently denoted by (JP)), we provide additional supporting simulations, an application of the MLPB to German stock index data and proofs omitted in the paper.

## 1. ADDITIONAL SIMULATION STUDIES

Analogue to the simulation study conducted in Section 6.2 of the paper, we apply MLPB, ARsieve, MBB and TBB to time series data of length $n = 100, 200, 500$ from an i.i.d. white noise process $\underline{X}_t = \underline{e}_t$ (WN model), where $\underline{e}_t \sim \mathcal{N}(0, \mathbf{\Sigma})$ is i.i.d. normally distributed. Note that here all proposed techniques remain valid for all choices of (sufficiently small) tuning parameters.

In Figure 1 and 2, the data is i.i.d. and in fact a bootstrap for dependent data is redundant to capture any dependence structure. However, we compare MLPB, ARsieve, MBB and TBB for $l, p, s = 1, 2, \ldots, 20$, i.e. the data generating process is over-fitted and hence slightly misspecified by MLPB and ARsieve. Observe that $\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l}$ computed for the MLPB becomes block-diagonal for $l < 0.5$ and the scheme degenerates to become an i.i.d. bootstrap as is true for the ARsieve if $p = 0$ [compare Remark 3.1], which would be of course most appropriate in this case, but are not reported here. This is in contrast to MBB and TBB, which simplify to become Efron's bootstrap for $s = 1$. However, the MLPB with data-adaptively selected banding parameters performs very well, where the individual choice seems to be slightly superior. The ARsieve appears to have problems in choosing the proper lag order $p = 0$ which leads to a larger RMSE, but does affect only slightly the performance wrt CR. In both Figures 1 and 2, it can be seen that wrt to RMSE and CR, all bootstrap procedures lose in terms of efficiency with redundantly increasing tuning parameters.

Figures 3 and 4 relate to Figures 1 and 2 in (JP) and show the corresponding results for the second coordinate of $\underline{\overline{X}}$ and $\underline{\mu}$, respectively.

## 2. A REAL DATA EXAMPLE

In this section, we apply the MLPB method to real data. We consider log-returns of stock prices of all 30 major German companies that are contained in the German Stock Index DAX (Deutscher Aktienindex) from July 1st, 2008 – January 11th, 2010. The exact constitution of the DAX is displayed in alphabetical order in Table 1 and the log-returns of the selected companies Allianz, Bayer, Deutsche Bank, EON, Metro, Siemens, Telekom and Volkswagen are shown in Figure 5.

Let $\underline{P}_t = (P_{1,t}, \ldots, P_{30,t})^T$ denote the vector of stock prices of the DAX companies at time $t$ ordered as in Table 1 and define $\underline{X}_t = (X_{1,t}, \ldots, X_{30,t})^T$ as the vector of their log-returns,

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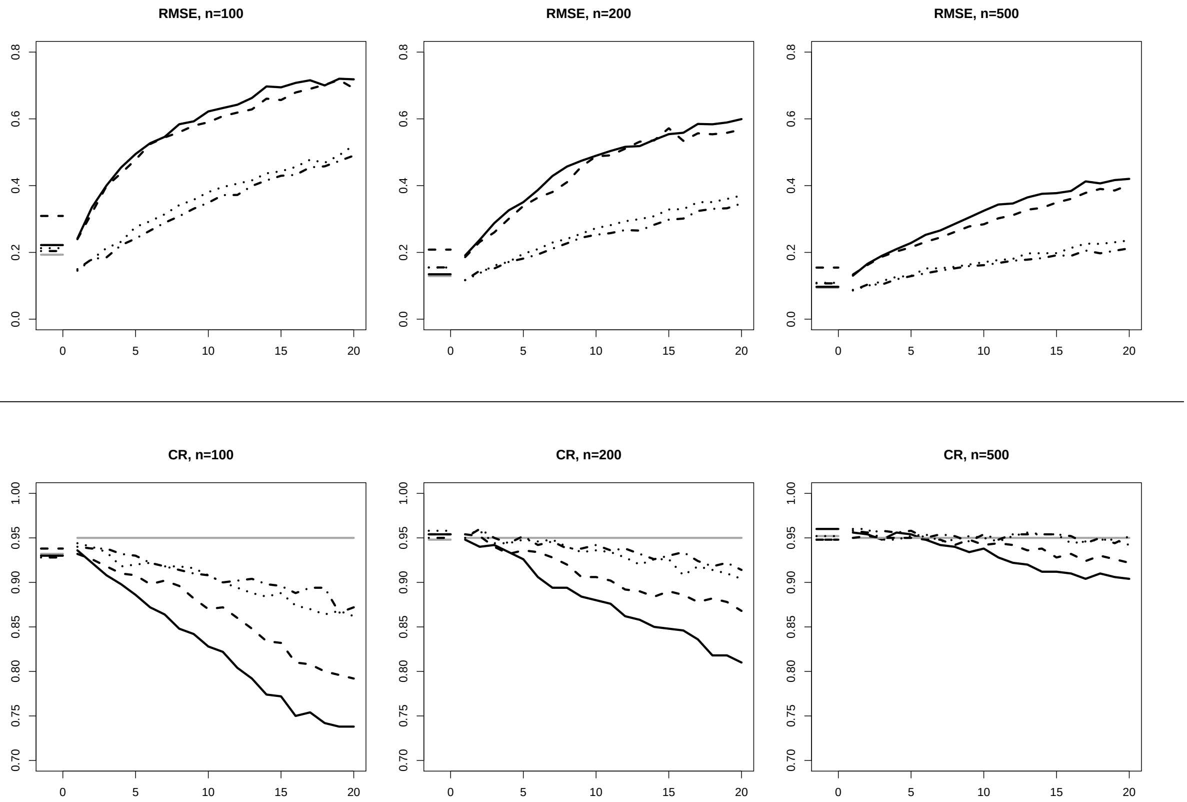


FIGURE 1. RMSE for estimating $Var(\sqrt{n}(\overline{X}_1 - \mu_1))$ and CR of bootstrap confidence intervals for $\mu_1$ by MLPB (solid), ARsieve (dashed), MBB (dotted) and TBB (dash-dotted) are reported vs. the respective tuning parameters $l, p, s \in \{1, \ldots, 20\}$ for the WN model with sample size $n \in \{100, 200, 500\}$. Line segments indicate results for data-adaptively chosen tuning parameters. MLPB with individual (grey) and global (black) banding parameter choice are reported.

i.e. $X_{j,t} = \log(P_{j,t}) - \log(P_{j,t-1})$. We consider log-return data for $d = 10, 20, 30$ stocks and $n = 100, 200, 300, 400$ trading days starting on July 1st, 2008. The MLPB procedure as described in Section 3 is applied to estimate the distribution of the average log-returns of equally weighted portfolios of these stocks over different time horizons. Based on an MLPB bootstrap sample, we compute its vector-valued sample mean and its weighted sum, where the weights are chosen to be equal and summing to one, i.e. in the notation of Section 5, we use the weight vector $\underline{b} = \frac{1}{d}1_d$. The case of equal weights is meant to mimic the realistic scenario of an investor having a fixed amount of money to be invested equally on the $d$ stocks chosen; increasing $d$ would then mean increasing the diversification of the investor's portfolio without increasing the total amount of money invested.

The via MLPB estimated standard deviations and sample means of the average log-returns of the portfolios are displayed in Table 2 and the corresponding densities obtained by the **R** function *density()* are shown in Figure 6, where the bandwidth is chosen by Silverman's rule-of-thumb and a Gaussian kernel has been used (the default setting of *density()* in **R**). To get these results, the banding parameter $l$ has been selected by the (global) rule suggested in Section 2.3

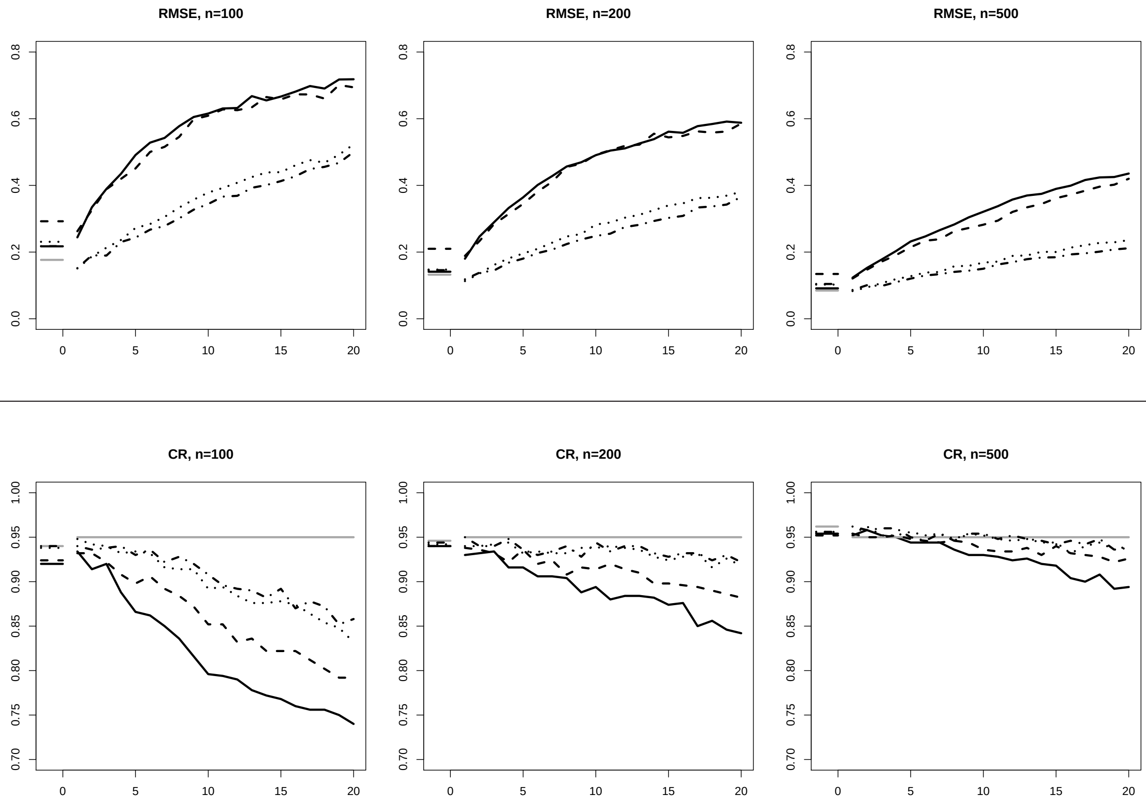


FIGURE 2. RMSE for estimating $Var(\sqrt{n}(\overline{X}_2 - \mu_2))$ and CR of bootstrap confidence intervals for $\mu_2$ by MLPB (solid), ARsieve (dashed), MBB (dotted) and TBB (dash-dotted) are reported vs. the respective tuning parameters $l, p, s \in \{1, \dots, 20\}$ for the WN model with sample size $n \in \{100, 200, 500\}$. Line segments indicate results for data-adaptively chosen tuning parameters. MLPB with individual (grey) and global (black) banding parameter choice are reported.

as displayed in Table 3, $B = 1000$ bootstrap replications have been generated and the tuning parameters $\epsilon$ and $\beta$ are set equal to one.

Going from top to bottom in Figure 6, the effect of diversification can be seen as the number of stocks in the portfolio is increasing; the apparent result is that the estimated distributions become less skewed and more normal-looking. Furthermore, increased diversification is associated with a decrease of the corresponding standard deviations as displayed in Table 2. Going from left to right in Figure 6, i.e., increasing sample size, the distributions become also more peaked due to the expected decrease in variance—see also Table 2. Although the rule for banding parameter selection seems to choose too large values for $l$ as shown in Table 3, the MLPB leads clearly to reasonable results. Table 3 also indicates that it may happen in practice that the selected banding parameter do not increase monotonically with sample size; this is, of course, a finite sample phenomenon.

To conclude the discussion of the real data example, two remarks are in order. First note

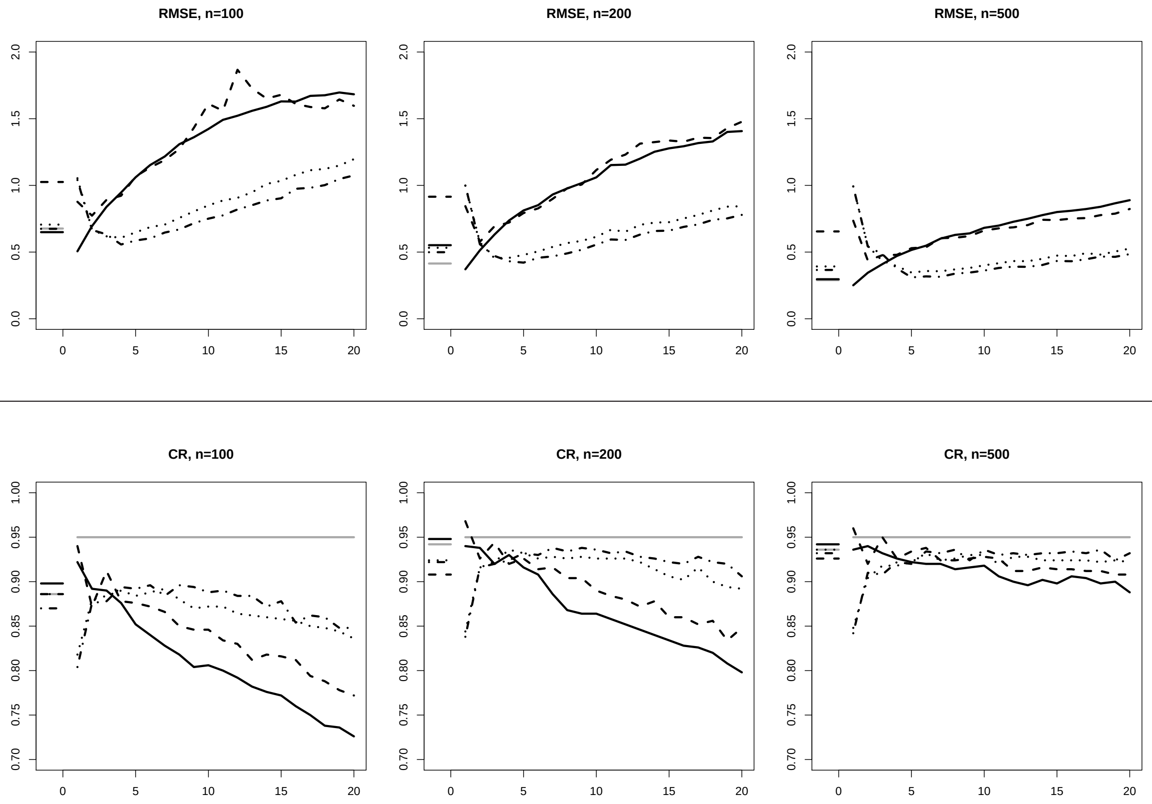


FIGURE 3. RMSE for estimating $Var(\sqrt{n}(\overline{X}_2 - \mu_2))$ and CR of bootstrap confidence intervals for $\mu_2$ by MLPB (solid), ARsieve (dashed), MBB (dotted) and TBB (dash-dotted) are reported vs. the respective tuning parameters $l, p, s \in \{1, \ldots, 20\}$ for the VMA(1) model with sample size $n \in \{100, 200, 500\}$. Line segments indicate results for data-adaptively chosen tuning parameters. MLPB with individual (grey) and global (black) banding parameter choice are reported.

that financial returns are invariably assumed to be nonlinear time series, and are typically modelled as ARCH or GARCH processes. However, the MLPB is proven to work also for the sample mean of time series that are not necessarily linear—see e.g. Theorem 4.1, and thus is applicable here. In this real data example, we have used up to $d = 30$ stocks and sample size $n = 400$. This leads to a $(12,000 \times 12,000)$ covariance matrix whose eigenvalues and eigenvectors have to be computed and which has to be Cholesky decomposed and inverted to apply the MLPB scheme. The simulations have been executed on the bwGrid Cluster provided for scientific computing by the University of Mannheim with **R** version 2.13.2, where it is possible to do this for $dn = 12,000$. Without having access to such computational power, we propose to use the less demanding modified bootstrap scheme described in Section 5.4 of the paper, which results in qualitatively similar results, if $S$ is chosen sufficiently small.

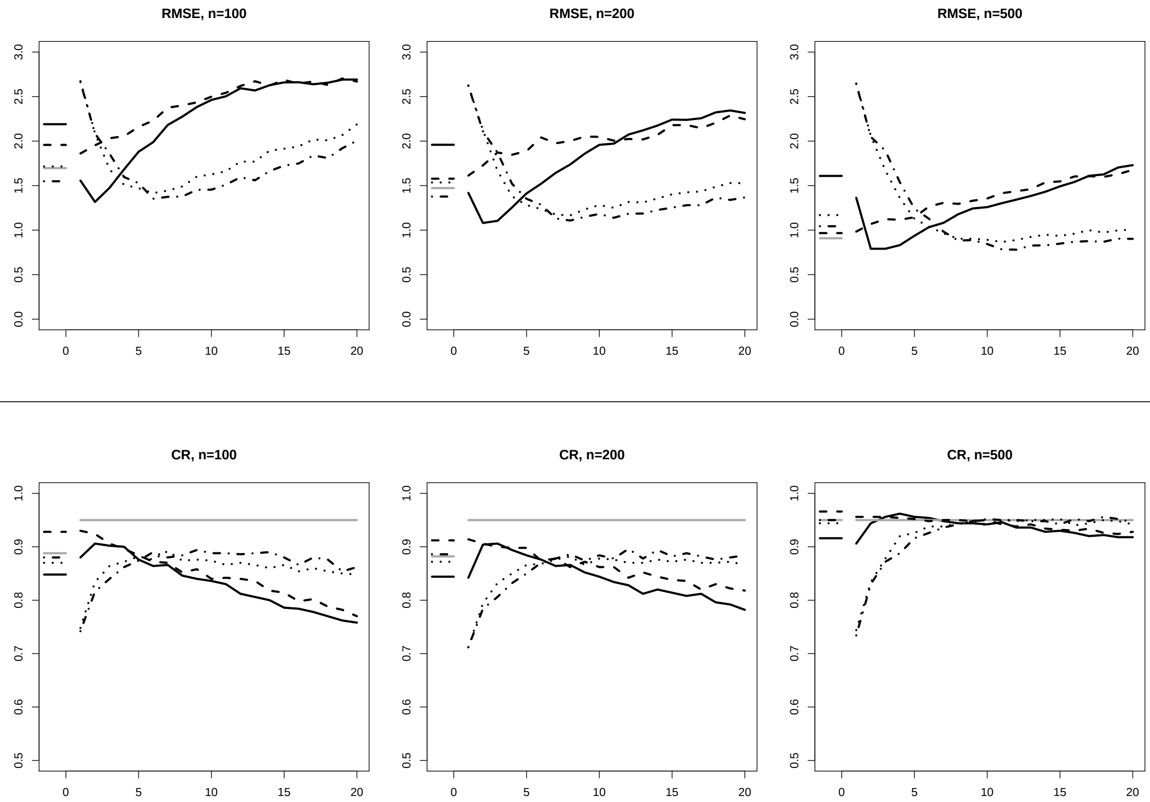


FIGURE 4. RMSE for estimating $Var(\sqrt{n}(\overline{X}_2 - \mu_2))$ and CR of bootstrap confidence intervals for $\mu_2$ by MLPB (solid), ARsieve (dashed), MBB (dotted) and TBB (dash-dotted) are reported vs. the respective tuning parameters $l, p, s \in \{1, \ldots, 20\}$ for the VAR(1) model with sample size $n \in \{100, 200, 500\}$. Line segments indicate results for data-adaptively chosen tuning parameters. MLPB with individual (grey) and global (black) banding parameter choice are reported.

| | | |
|---|---|---|
| Adidas | Deutsche Lufthansa | Linde |
| Allianz | Deutsche Post | MAN |
| BASF | Deutsche Telekom | Merck KGAA |
| Bayer | E.ON | Metro |
| Beiersdorf | Fresenius | Münchner Rück |
| BMW | Fresenius Medical Care | RWE |
| Commerzbank | Heidelberg Cement | SAP |
| Daimler | Henkel | Siemens |
| Deutsche Bank | Infineon | Thyssen Krupp |
| Deutsche Börse | K+S | Volkswagen |

TABLE 1. DAX companies (on June 1st, 2012).

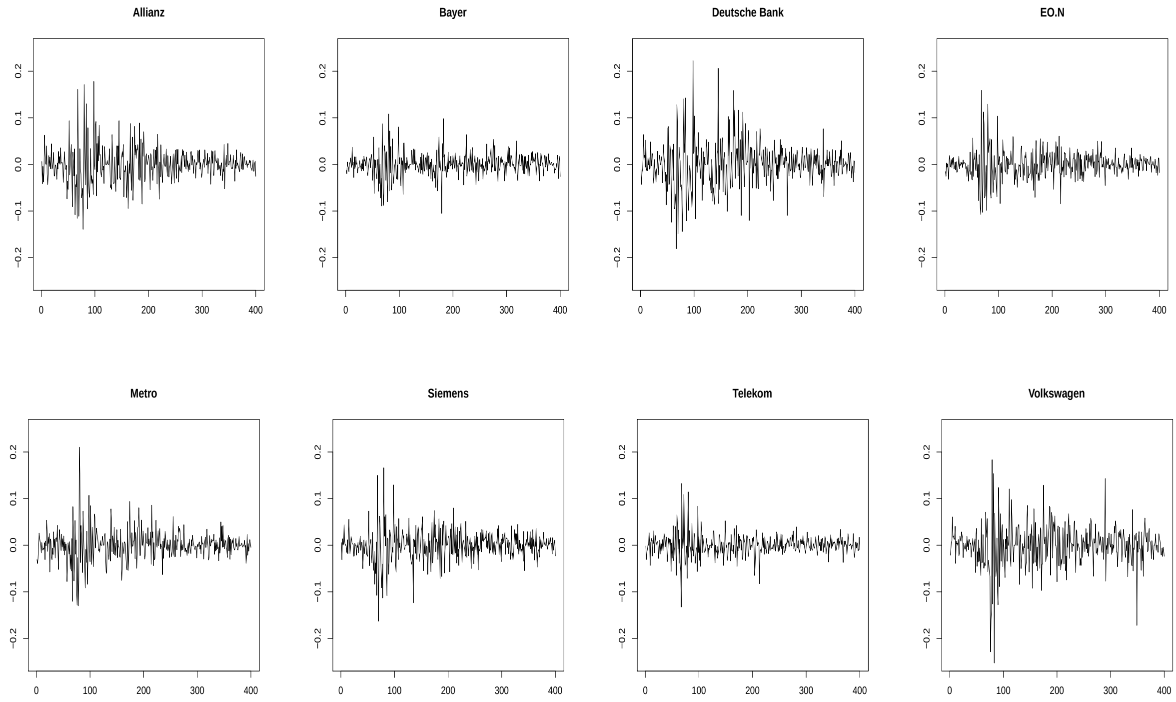


FIGURE 5. Log-returns of stock prices of selected German companies contained in the German stock index DAX for 400 trading days starting on July 1st, 2008.

| | $n = 100$ | | $n = 200$ | | $n = 300$ | | $n = 400$ | |
|---|---|---|---|---|---|---|---|---|
| d | mean | sd | mean | sd | mean | sd | mean | sd |
| 10 | -0.1309 | 1.1227 | -0.4461 | 0.8266 | -0.1449 | 0.7167 | 0.6661 | 0.5414 |
| 20 | -0.8819 | 0.7957 | 0.2131 | 0.6524 | -0.0601 | 0.5120 | 0.1505 | 0.5206 |
| 30 | -0.7321 | 0.8084 | 0.7269 | 0.6058 | -0.0511 | 0.5790 | 0.0536 | 0.4911 |

TABLE 2. Comparison of bootstrap standard deviations and sample means $\times 10^3$, respectively, of average log-returns of equally weighted portfolios for different number of stocks $d = 10, 20, 30$ and $n = 100, 200, 300, 400$ trading days starting on July 1st, 2008.

| d \ n | 100 | 200 | 300 | 400 |
|---|---|---|---|---|
| 10 | 18 | 14 | 16 | 16 |
| 20 | 18 | 19 | 36 | 36 |
| 30 | 28 | 37 | 36 | 37 |

TABLE 3. Banding parameters selected by the rule-of-thumb suggested in Section 2.3 for log-return data for different number of stocks $d = 10, 20, 30$ and $n = 100, 200, 300, 400$ trading days starting on July 1st, 2008.

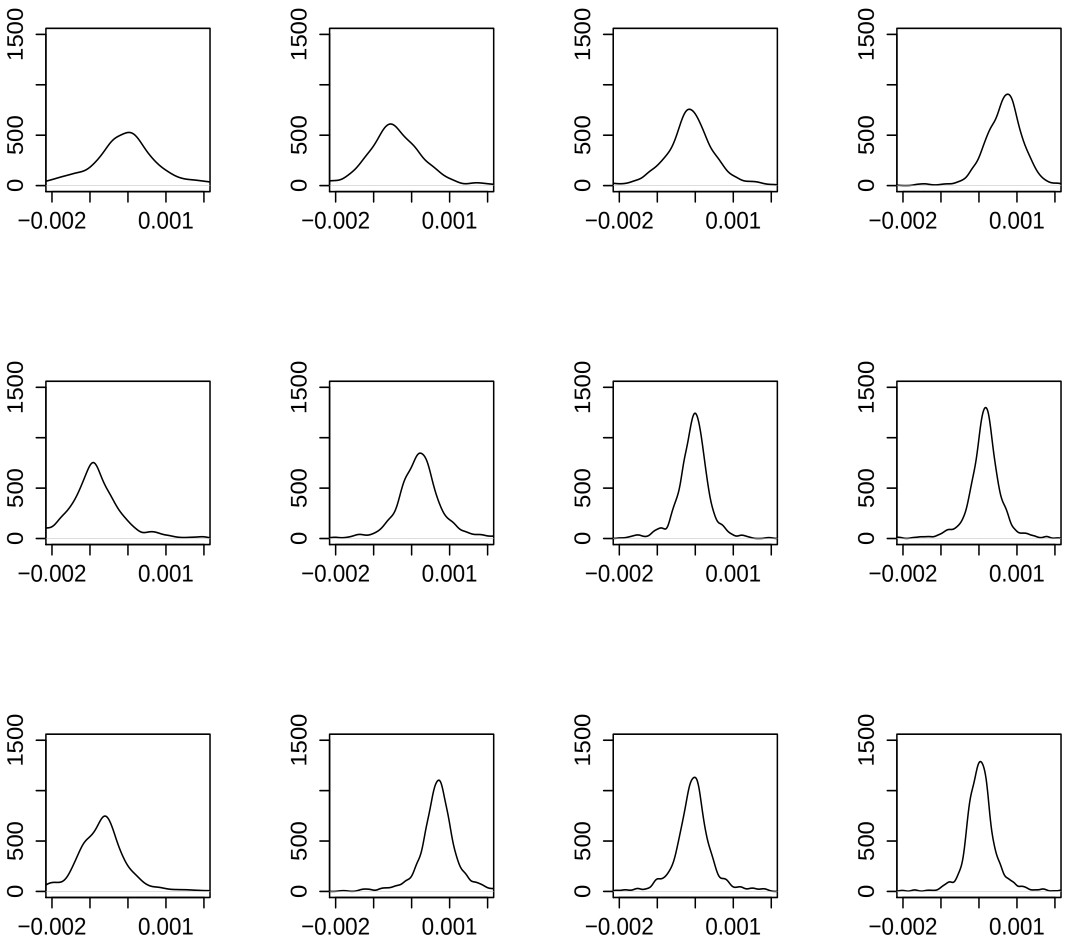


FIGURE 6. Bootstrap density plots of the average log-returns of an equally weighted portfolio that contains $d = 10, 20, 30$ stocks (from top to bottom) for $n = 100, 200, 300, 400$ (from left to right) trading days starting on July 1st, 2008.

## APPENDIX A. PROOFS

**Proof of Theorem 2.1.** Symmetry of $\widehat{\mathbf{\Gamma}}_{\kappa,l} - \mathbf{\Gamma}_{dn}$ together with problem 21, p. 313 in Horn and Johnson (1990) and plugging-in for $\widehat{\Gamma}_{\kappa,l}(i,j)$ and $\Gamma_{dn}(i,j)$ yields

$$\begin{aligned}\rho(\widehat{\mathbf{\Gamma}}_{\kappa,l} - \mathbf{\Gamma}_{dn}) &\leq \max_{1\leq j\leq dn} \sum_{i=1}^{dn} |\kappa_l(m_2(i,j))\widehat{C}_{\underline{m}_1(i,j)}(m_2(i,j)) - C_{\underline{m}_1(i,j)}(m_2(i,j))| \\ &\leq 2\sum_{i,j=1}^{d}\sum_{h=0}^{n-1} |\kappa_l(h)\widehat{C}_{ij}(h) - C_{ij}(h)|,\end{aligned}$$

where the second inequality is implied by the definitions of $\underline{m}_1(\cdot,\cdot)$ and $m_2(\cdot,\cdot)$ in (2.4). By splitting-up the expression on the last right-hand side above and plugging-in for $\kappa_l(h)$, we get

$$\begin{aligned}\rho(\widehat{\mathbf{\Gamma}}_{\kappa,l}-\mathbf{\Gamma}_{dn}) &\leq 2\sum_{i,j=1}^{d}\left(\sum_{h=0}^{l}|\widehat{C}_{ij}(h)-C_{ij}(h)|+\sum_{h=l+1}^{\lfloor c_\kappa l\rfloor}|\kappa_l(h)\widehat{C}_{ij}(h)-C_{ij}(h)|+\sum_{h=\lfloor c_\kappa l\rfloor+1}^{n-1}|C_{ij}(h)|\right)\\ &=: A_1+A_2+A_3.\end{aligned}$$

Now, before considering $A_1$ and $A_2$ separately, note that

$$\begin{aligned}\|\widehat{C}_{ij}(h)-C_{ij}(h)\|_2 &\leq \frac{1}{n}\|\sum_{t=max(1,1-h)}^{min(n,n-h)}(X_{i,t+h}-\overline{X}_i)(X_{j,t}-\overline{X}_j)-(n-|h|)C_{ij}(h)\|_2+\frac{|h|}{n}\|C_{ij}(h)\|_2\\ &\leq \frac{M}{\sqrt{n}}+\frac{|h|}{n}|C_{ij}(h)|, \qquad\qquad (A.1)\end{aligned}$$

where the second inequality follows from assumption (A2). Using this result, we obtain

$$\|A_1\|_2\leq 2\sum_{i,j=1}^{d}\sum_{h=0}^{l}\left(\frac{M}{\sqrt{n}}+\frac{|h|}{n}|C_{ij}(h)|\right)\leq\frac{2Md^2(l+1)}{\sqrt{n}}+2\sum_{h=0}^{l}\frac{|h|}{n}|\mathbf{C}(h)|_1.$$

Now, consider $A_2$. First, it holds

$$\begin{aligned}A_2 &\leq 2\sum_{i,j=1}^{d}\sum_{h=l+1}^{\lfloor c_\kappa l\rfloor}|\kappa_l(h)||\widehat{C}_{ij}(h)-C_{ij}(h)|+2\sum_{h=l+1}^{\lfloor c_\kappa l\rfloor}|\kappa_l(h)-1||\mathbf{C}(h)|_1\\ &\leq 2\sum_{i,j=1}^{d}\sum_{h=l+1}^{\lfloor c_\kappa l\rfloor}|\widehat{C}_{ij}(h)-C_{ij}(h)|+2\sum_{h=l+1}^{\lfloor c_\kappa l\rfloor}|\mathbf{C}(h)|_1\end{aligned}$$

and straightforward application of (A.1) results in

$$\begin{aligned}\|A_2\|_2 &\leq 2\sum_{i,j=1}^{d}\sum_{h=l+1}^{\lfloor c_\kappa l\rfloor}\left(\frac{M}{\sqrt{n}}+\frac{|h|}{n}|C_{ij}(h)|\right)+2\sum_{h=l+1}^{\lfloor c_\kappa l\rfloor}|\mathbf{C}(h)|_1\\ &\leq \frac{2Md^2(\lfloor c_\kappa l\rfloor-l)}{\sqrt{N}}+2\sum_{h=l+1}^{\lfloor c_\kappa l\rfloor}\frac{|h|}{n}|\mathbf{C}(h)|_1+2\sum_{h=l+1}^{\lfloor c_\kappa l\rfloor}|\mathbf{C}(h)|_1.\end{aligned}$$

□

**Proof of Theorem 2.2.**
Without loss of generality, the eigenvalues of $\widehat{\mathbf{R}}_{\kappa,l}$ can be ordered such that $r_1\geq r_2\geq\cdots\geq r_{dn}$ and let $\lambda_{max}(\mathbf{A})$ and $\lambda_{min}(\mathbf{A})$ denote the largest and the smallest eigenvalue of a matrix $\mathbf{A}$, respectively. Then, by Corollary 4.3.3 in Horn and Johnson (1990), it holds

$$-r_{dn}=-\lambda_{min}(\widehat{\mathbf{R}}_{\kappa,l})=\lambda_{max}(-\widehat{\mathbf{R}}_{\kappa,l})\leq\lambda_{max}(\mathbf{R}_{dn}-\widehat{\mathbf{R}}_{\kappa,l})\leq\rho(\mathbf{R}_{dn}-\widehat{\mathbf{R}}_{\kappa,l}), \qquad (A.2)$$

where $\mathbf{R}_{dn}=\widehat{\mathbf{V}}^{-1/2}\mathbf{\Gamma}_{dn}\widehat{\mathbf{V}}^{-1/2}$ and the last inequality follows from the definition of the operator norm and the spectral factorization of a symmetric matrix. Further, it holds

$$\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l}-\widehat{\mathbf{\Gamma}}_{\kappa,l}=\widehat{\mathbf{V}}^{1/2}\mathbf{S}(\mathbf{D}^{\epsilon}-\mathbf{D})\mathbf{S}^T\widehat{\mathbf{V}}^{1/2},$$

where

$$\mathbf{D}^{\epsilon}-\mathbf{D}=diag\left(max(r_i,\epsilon n^{-\beta})-r_i,i=1,\ldots,dn\right)=diag\left(max(0,\epsilon n^{-\beta}-r_i),i=1,\ldots,dn\right).$$

Together with (A.2) and

$$\rho(\mathbf{\Gamma}_{dn} - \widehat{\mathbf{\Gamma}}_{\kappa,l}) \leq \rho^2(\mathbf{V}^{1/2})\rho(\mathbf{S}(\mathbf{D}^{\epsilon} - \mathbf{D})\mathbf{S}^T) = \max_i \widehat{C}_{ii}(0) \cdot \max(0, \epsilon n^{-\beta} - r_{dn})$$

this leads to

$$\begin{aligned} \rho(\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l} - \mathbf{\Gamma}_{dn}) &\leq max(0, \epsilon\ max_i\ \widehat{C}_{ii}(0)n^{-\beta} - r_{dn}) + \rho(\widehat{\mathbf{\Gamma}}_{\kappa,l} - \mathbf{\Gamma}_{dn}) \\ &\leq \max_i \widehat{C}_{ii}(0) \left(\epsilon n^{-\beta} + \frac{1}{\max_i \widehat{C}_{ii}(0)} \rho(\mathbf{\Gamma}_{dn} - \widehat{\mathbf{\Gamma}}_{\kappa,l})\right) + \rho(\mathbf{\Gamma}_{dn} - \widehat{\mathbf{\Gamma}}_{\kappa,l}) \\ &= \epsilon\ \max_i\ \widehat{C}_{ii}(0)n^{-\beta} + 2\rho(\widehat{\mathbf{\Gamma}}_{\kappa,l} - \mathbf{\Gamma}_{dn}). \end{aligned}$$

From $\|\widehat{C}_{ii}(0)\|_2 = C_{ii}(0) + O(\frac{1}{\sqrt{n}})$, $i = 1, \ldots, d$ and Theorem 2.1, we get the desired result. □

**Proof of Corollary 2.1.**
By (A1), Gerschgorins Theorem and by (A3), we have that $\rho(\mathbf{\Gamma}_{dn})$, $\rho(\mathbf{\Gamma}_{dn}^{-1})$, $\rho(\mathbf{\Gamma}_{dn}^{1/2})$ and $\rho(\mathbf{\Gamma}_{dn}^{-1/2})$ are bounded from above and from below. The claimed result for $\rho(\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l} - \mathbf{\Gamma}_{dn})$ follows directly from Theorem 2.2and the corresponding result for the inverses of $\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l}$ and $\mathbf{\Gamma}_{dn}$ follows from the proof of Theorem 2 in McMurry and Politis (2010). The claimed convergence of the Cholesky matrices is established through Theorem 2.1 of Drmac, Omladic and Veselic (1994), which provides the bound

$$\rho\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{1/2} - \mathbf{\Gamma}_{dn}^{1/2}\right) \leq \frac{2c_n\rho(\mathbf{\Gamma}_{dn}^{1/2})\rho(\mathbf{\Gamma}_{dn}^{-1/2}(\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l} - \mathbf{\Gamma}_{dn})(\mathbf{\Gamma}_{dn}^{-1/2})^T)}{\sqrt{1 - 4c_n^2\rho(\mathbf{\Gamma}_{dn}^{-1/2}(\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l} - \mathbf{\Gamma}_{dn})(\mathbf{\Gamma}_{dn}^{-1/2})^T)}}, \tag{A.3}$$

where $c_n = \frac{1}{2} + \lfloor \log_2(dn) \rfloor$ if the radicand in the denominator above is strictly positive. Since $\rho(\mathbf{\Gamma}_{dn}^{1/2})$ and $\rho(\mathbf{\Gamma}_{dn}^{-1/2})$ are bounded from above, the desired results hold if $\log^2(n) \cdot r_{l,n} = o(1)$. The corresponding result for their inverses follows from

$$\mathbf{\Gamma}_{dn}^{-1/2} - (\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2} = (\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{1/2} - \mathbf{\Gamma}_{dn}^{1/2}\right)\mathbf{\Gamma}_{dn}^{-1/2},$$

which also implies boundedness of $\rho((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{1/2})$ and $\rho((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2})$ from above and from below, which concludes this proof. □

**Proof of Theorem 4.1.**
Let $\underline{\widetilde{Z}}^*$ be the bootstrap sample that is defined analogue to $\underline{Z}^*$ in Step 4 of Section 3 in (JP) except that the resample is drawn from the standardized values of $\underline{\widetilde{W}} = \mathbf{\Gamma}_{dn}^{-1/2}\underline{Y}$ instead of $\underline{W} = (\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}\underline{Y}$. Now, the proof of Theorem 4.1 proceeds through a sequence of lemmas. Lemma A.1(i) gives the justification for using

$$\underline{\widetilde{Y}}^* = \mathbf{\Gamma}_{dn}^{1/2}\underline{\widetilde{Z}}^* \tag{A.4}$$

in all subsequent computations instead of $\underline{Y}^*$. Furthermore, we define $\mathbf{C}^{(k)}(h) = \mathbf{C}(h)1(|h| \leq k)$ and let $\mathbf{\Gamma}_{dn,k} = (\mathbf{C}^{(k)}(i-j), i, j = 1, \ldots, n)$ be the $k$-banded version of $\mathbf{\Gamma}_{dn}$. The matrix $\mathbf{\Gamma}_{dn,k}$ is banded in the sense that only the $(2k+1)d$ main diagonals are not equal to zero and for all sequences $k = k(n) \to \infty$, it holds $\rho(\mathbf{\Gamma}_{dn,k} - \mathbf{\Gamma}_{dn}) \to 0$ as $n \to \infty$ due to $\rho(\mathbf{\Gamma}_{dn,k} - \mathbf{\Gamma}_{dn}) \leq 2\sum_{h=k+1}^{\infty} |\mathbf{C}(h)|_1$, which is obtained analogue to the proof of Theorem 2.1. Let $\mathbf{\Gamma}_{dn,k}^{1/2}$ be the Cholesky decomposition of $\mathbf{\Gamma}_{dn,k}$ which exists for sufficiently large $k$ by (A3) and note that only its main diagonal and the $d(k+1)$ secondary diagonals below the main diagonal contain non zero elements which are all bounded by $max_i\ C_{ii}^{1/2}(0)$. The second part of Lemma A.1 allows us also to replace $\mathbf{\Gamma}_{dn}^{1/2}$ in (A.4) by $\mathbf{\Gamma}_{dn,k}^{1/2}$ and to get asymptotically the same results. Lemma A.2 gives the proper limiting bootstrap variance, while Lemma A.3 proves boundedness in probability of $E^*(\widetilde{Z}_i^{*4})$ and Lemma A.4 deals with asymptotic normality of $\frac{1}{\sqrt{n}}\sum_{t=1}^n \underline{\widetilde{Y}}_t^*$.

**Lemma A.1.** *Under the assumptions of Theorem 4.1, it holds*

$$(i) \quad \frac{1}{\sqrt{n}}\sum_{t=1}^n \underline{Y}_t^* - \frac{1}{\sqrt{n}}\sum_{t=1}^n \underline{\widetilde{Y}}_t^* = o_{P^*}(1) \quad \text{and} \quad (ii) \quad \frac{1}{\sqrt{n}}\sum_{t=1}^n \underline{\widetilde{Y}}_t^* - \frac{1}{\sqrt{n}}\sum_{t=1}^n \underline{\widetilde{Y}}_t^{*,k} = o_{P^*}(1),$$

*where* $\underline{\widetilde{Y}}^{*,k} = \mathbf{\Gamma}_{dn,k}^{1/2}\underline{\widetilde{Z}}^*$.

*Proof.* (i) Let $\mathbf{J} = [\mathbf{I}_d : \cdots : \mathbf{I}_d]$ be the $(d \times dn)$ matrix that consists of $n$ $(d \times d)$ unit matrices. In the following, we show that $\frac{1}{\sqrt{n}}\underline{b}^T\sum_{t=1}^n \underline{Y}_t^* = \frac{1}{\sqrt{n}}\underline{b}^T\sum_{t=1}^n \underline{\widetilde{Y}}_t^* + O_{P^*}(\log^2(n)\cdot r_{l,n})$ for any $\mathbb{R}^d$-valued vector $\underline{b}$. We have

$$\begin{aligned}
\frac{1}{\sqrt{n}}\underline{b}^T\sum_{t=1}^n \underline{Y}_t^* &= \frac{1}{\sqrt{n}}\underline{b}^T\mathbf{J}(\mathbf{\Gamma}_{dn})^{1/2}\underline{\widetilde{Z}}^* + \frac{1}{\sqrt{n}}\underline{b}^T\mathbf{J}\mathbf{\Gamma}_{dn}^{1/2}(\underline{Z}^* - \underline{\widetilde{Z}}^*) + \frac{1}{\sqrt{n}}\underline{b}^T\mathbf{J}\left((\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{1/2} - \mathbf{\Gamma}_{dn}^{1/2}\right)\underline{Z}^* \\
&= \frac{1}{\sqrt{n}}\underline{b}^T\sum_{t=1}^n \underline{\widetilde{Y}}_t^* + R_1^* + R_2^*
\end{aligned}$$

and it remains to show that $R_1^*$ and $R_2^*$ are asymptotically negligible. Considering $R_2^*$, we get

$$\begin{aligned}
E^*(R_2^{*2}) &= \frac{1}{n}\underline{b}^T\mathbf{J}((\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{1/2} - \mathbf{\Gamma}_{dn}^{1/2})((\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{1/2} - \mathbf{\Gamma}_{dn}^{1/2})^T\mathbf{J}^T\underline{b} \\
&\leq \frac{1}{n}\underline{b}^T\mathbf{J}\mathbf{J}^T\underline{b}\lambda_{max}\left(\left((\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{1/2} - \mathbf{\Gamma}_{dn}^{1/2}\right)\left((\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{1/2} - \mathbf{\Gamma}_{dn}^{1/2}\right)^T\right) \\
&= |\underline{b}|_2^2\rho^2\left((\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{1/2} - \mathbf{\Gamma}_{dn}^{1/2}\right)
\end{aligned}$$

and this leads to $R_2^* = O_{P^*}(\log^2(n)\cdot r_{l,n})$ by Corollary 2.1. Now, we turn to $R_1^*$. First of all, observe that $\underline{Z}^*$ and $\underline{\widetilde{Z}}^*$ may be represented as

$$\begin{aligned}
\underline{Z}^* &= \mathbf{M}^*\frac{1}{\widehat{\sigma}_W}\left(\mathbf{I}_{dn} - \frac{1}{dn}\mathbf{1}_{dn\times dn}\right)(\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{-1/2}\underline{Y}, \\
\underline{\widetilde{Z}}^* &= \mathbf{M}^*\frac{1}{\widehat{\sigma}_{\widetilde{W}}}\left(\mathbf{I}_{dn} - \frac{1}{dn}\mathbf{1}_{dn\times dn}\right)(\mathbf{\Gamma}_{dn})^{-1/2}\underline{Y},
\end{aligned}$$

where $\mathbf{I}_{dn}$ is the $(dn \times dn)$ unit matrix, $\mathbf{1}_{dn\times dn}$ is the $(dn \times dn)$ matrix of ones and each row of the $(dn \times dn)$ matrix $\mathbf{M}^*$ is independently and uniformly selected from the standard basis vectors $e_1, \ldots, e_{dn}$. This yields

$$\begin{aligned}
R_1^* &= \frac{1}{\widehat{\sigma}_{\widetilde{W}}}\frac{1}{\sqrt{n}}\underline{b}^T\mathbf{J}\mathbf{\Gamma}_{dn}^{1/2}\mathbf{M}^*\left(\mathbf{I}_{dn} - \frac{1}{dn}\mathbf{1}_{dn\times dn}\right)\left((\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{-1/2} - (\mathbf{\Gamma}_{dn})^{-1/2}\right)\underline{Y} \\
&\quad + \left(\frac{1}{\widehat{\sigma}_W} - \frac{1}{\widehat{\sigma}_{\widetilde{W}}}\right)\frac{1}{\sqrt{n}}\underline{b}^T\mathbf{J}\mathbf{\Gamma}_{dn}^{1/2}\mathbf{M}^*\left(\mathbf{I}_{dn} - \frac{1}{dn}\mathbf{1}_{dn\times dn}\right)(\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{-1/2}\underline{Y} \\
&= R_3^* + R_4^*
\end{aligned}$$

and for $R_3^*$, we get

$$
\begin{aligned}
E^*(R_3^{*2}) &= E^*\left(\frac{1}{\widehat{\sigma}^2_{\widetilde{W}}}\frac{1}{n}\underline{b}^T\mathbf{J}\mathbf{\Gamma}_{dn}^{1/2}\mathbf{M}^*\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}-(\mathbf{\Gamma}_{dn})^{-1/2}\right)\underline{Y}\right.\\
&\qquad \left.\times\underline{Y}^T\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}-(\mathbf{\Gamma}_{dn})^{-1/2}\right)^T\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)^T\mathbf{M}^{*T}(\mathbf{\Gamma}_{dn}^{1/2})^T\mathbf{J}^T\underline{b}\right)\\
&= \frac{1}{\widehat{\sigma}^2_{\widetilde{W}}}\frac{1}{n}\underline{b}^T\mathbf{J}\mathbf{\Gamma}_{dn}^{1/2}E^*\left(\underline{V}^*\underline{V}^{*T}\right)(\mathbf{\Gamma}_{dn}^{1/2})^T\mathbf{J}^T\underline{b},
\end{aligned}
$$

where $\underline{V}^*$ is a bootstrap vector drawn from $(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn})\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}-(\mathbf{\Gamma}_{dn})^{-1/2}\right)\underline{Y}$. Due to independent resampling, it holds $E^*\left(\underline{V}^*\underline{V}^{*T}\right)=\sigma_V^2\mathbf{I}_{dn}$ with

$$
\begin{aligned}
\sigma_V^2 &= \underline{Y}^T\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}-(\mathbf{\Gamma}_{dn})^{-1/2}\right)^T\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)^T\\
&\qquad \times E^*(M_{j\bullet}^{*T}M_{j\bullet}^*)\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}-(\mathbf{\Gamma}_{dn})^{-1/2}\right)\underline{Y}\\
&= \frac{1}{dn}\underline{Y}^T\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}-(\mathbf{\Gamma}_{dn})^{-1/2}\right)^T\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)^2\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}-(\mathbf{\Gamma}_{dn})^{-1/2}\right)\underline{Y}\\
&\leq \frac{1}{dn}\underline{Y}^T\underline{Y}\lambda_{max}\left(\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}-(\mathbf{\Gamma}_{dn})^{-1/2}\right)^T\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)^2\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}-(\mathbf{\Gamma}_{dn})^{-1/2}\right)\right)\\
&\leq \frac{1}{dn}\underline{Y}^T\underline{Y}\rho^2\left(\left((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}-(\mathbf{\Gamma}_{dn})^{-1/2}\right)^T\right)\rho^2\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right),
\end{aligned}
$$

where $M_{j\bullet}^*$ denotes the $j$th row of $\mathbf{M}^*$ and $E^*(M_{j\bullet}^{*T}M_{j\bullet}^*)=\frac{1}{dn}\mathbf{I}_{dn}$. Thanks to $\underline{Y}^T\underline{Y}=O_P(dn)$, $\rho(((\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}-(\mathbf{\Gamma}_{dn})^{-1/2})^T)=O_P(\log^2(n)\cdot r_{l,n})$ and $\rho\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)=O(1)$, we obtain

$$
E^*(R_3^{*2})\leq\frac{\sigma_V^2}{\widehat{\sigma}^2_{\widetilde{W}}}\frac{1}{n}\underline{b}^T\mathbf{J}\mathbf{J}^T\underline{b}\lambda_{max}(\mathbf{\Gamma}_{dn})=\frac{\sigma_V^2}{\widehat{\sigma}^2_{\widetilde{W}}}|\underline{b}|_2^2\rho^2(\mathbf{\Gamma}_{dn}^{1/2})
$$

resulting in $R_3^*=O_{P^*}(\log^2(n)\cdot r_{l,n})$ because $\widehat{\sigma}^2_{\widetilde{W}}$ is bounded away from zero and from above with probability tending to one. Finally, since the same holds true for $\widehat{\sigma}^2_W$, to handle $R_4^*$, it is sufficient to show $|\widehat{\sigma}^2_W-\widehat{\sigma}^2_{\widetilde{W}}|=O_P(\log^2(n)\cdot r_{l,n})$. By plugging-in, we have

$$
\begin{aligned}
|\widehat{\sigma}^2_{\widetilde{W}}-\widehat{\sigma}^2_W| &= \left|\frac{1}{dn}\underline{Y}^T(\mathbf{\Gamma}_{dn}^{-1/2})^T\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)^2\left(\mathbf{\Gamma}_{dn}^{-1/2}-(\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}\right)\underline{Y}\right|\\
&\qquad +\left|\frac{1}{dn}\underline{Y}^T\left(\mathbf{\Gamma}_{dn}^{-1/2}-(\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}\right)^T\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)^2(\widehat{\mathbf{\Gamma}}^{\epsilon}_{\kappa,l})^{-1/2}\underline{Y}\right|
\end{aligned}
$$

and by Cauchy-Schwarz inequality, the first term above is bounded by

$$\begin{aligned}
&\left(\frac{1}{dn}\underline{Y}^T(\mathbf{\Gamma}_{dn}^{-1/2})^T\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)^2(\mathbf{\Gamma}_{dn}^{-1/2})\underline{Y}\right)^{1/2} \\
&\times\left(\frac{1}{dn}\underline{Y}^T\left(\mathbf{\Gamma}_{dn}^{-1/2}-(\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{-1/2}\right)^T\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)^2\left(\mathbf{\Gamma}_{dn}^{-1/2}-(\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{-1/2}\right)\underline{Y}\right)^{1/2} \\
\leq\;& \frac{1}{dn}\underline{Y}^T\underline{Y}\rho\left((\mathbf{\Gamma}_{dn}^{-1/2})^T\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)\right)\rho\left(\left(\mathbf{\Gamma}_{dn}^{-1/2}-(\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{-1/2}\right)^T\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)\right) \\
=\;& O_P(\log^2(n)\cdot r_{l,n}).
\end{aligned}$$

Analogue computations for the second term leads to $R_4^*=O_{P^*}(\log^2(n)\cdot r_{l,n})$, which concludes the proof of the first assertion. The claimed equality in (ii) is obtained from a similar calculation as executed for $R_2^*$, where $\rho(\mathbf{\Gamma}_{dn,k}^{1/2}-\mathbf{\Gamma}_{dn}^{1/2})=O(\log^2(n)\cdot s_k)$ is used. The last result follows as in the proof of Corollary 2.2, from (A.3) with $(\widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon})^{1/2}$ replaced by $\mathbf{\Gamma}_{dn,k}^{1/2}$ and $\rho(\mathbf{\Gamma}_{dn,k}-\mathbf{\Gamma}_{dn})=O(s_k)$, where $s_k=\sum_{h=k+1}^{\infty}|\mathbf{C}(h)|_1$. □

**Lemma A.2.** *Under the assumptions of Theorem 4.1,* $Var^*(\frac{1}{\sqrt{n}}\sum_{t=1}^n\underline{\widetilde{Y}}_t^*)=\sum_{h\in\mathbb{Z}}\mathbf{C}(h)+o_P(1)$.

*Proof.* For any $\mathbb{R}^d$-valued vector $\underline{b}$, we get by standard arguments

$$Var^*\left(\underline{b}^T\frac{1}{\sqrt{n}}\sum_{t=1}^n\underline{\widetilde{Y}}_t^*\right)=\frac{1}{n}\underline{b}^T\mathbf{J}\mathbf{\Gamma}_{dn}\mathbf{J}^T\underline{b}=\underline{b}^T\left(\frac{1}{n}\sum_{i,j=1}^n\mathbf{C}(i-j)\right)\underline{b}=\underline{b}^T\left(\sum_{h=-\infty}^{\infty}\mathbf{C}(h)\right)\underline{b}+o(1).$$

□

**Lemma A.3.** *Under (A1)–(A4) with some $q\geq 2$, we have $E^*(|\widetilde{Z}_1|^{*q})=O_P(1)$. Under (A1′)–(A4′), we have $E^*(|\widetilde{Z}_1|^{*q})=O_P(d^{q/2})$.*

*Proof.* Due to $E^*(|\widetilde{Z}_1|^{*q})=\frac{1}{dn}\sum_{t=1}^{dn}|\widetilde{Z}_t|^q$, we show $E(|\widetilde{Z}_t|^q)=\|\widetilde{Z}_t\|_q^q=O(1)$ uniformly in $t$. By defining $(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn})(\mathbf{\Gamma}_{dn})^{-1/2}=(a_{ij})_{i,j=1,\ldots,dn}$ and $\underline{A}_t(j)=(a_{t,(j-1)d+1},\ldots,a_{t,jd})^T$, we get

$$\widetilde{Z}_t=\frac{1}{\widehat{\sigma}_{\widetilde{W}}}e_t^T\left(\mathbf{I}_{dn}-\frac{1}{dn}\mathbf{1}_{dn\times dn}\right)(\mathbf{\Gamma}_{dn})^{-1/2}\underline{Y}=\frac{1}{\widehat{\sigma}_{\widetilde{W}}}\left(\sum_{j=1}^n\underline{A}_t^T(j)(\underline{X}_j-\underline{\mu})-\sum_{j=1}^n\underline{A}_t^T(j)\left(\underline{\overline{X}}-\underline{\mu}\right)\right)$$

and thanks to boundedness of $\frac{1}{\widehat{\sigma}_{\widetilde{W}}}$, it suffices to consider the two terms in parentheses above in the following. For the second one, we get from triangle inequality and (A4) that

$$\|\sum_{j=1}^n\underline{A}_t^T(j)\left(\underline{\overline{X}}-\underline{\mu}\right)\|_q^q=\|\sum_{j=1}^n\sum_{s=1}^d a_{t,(j-1)d+s}\left(\overline{X}_s-\mu_s\right)\|_q^q\leq\left(\sum_{j=1}^n\sum_{s=1}^d|a_{t,(j-1)d+s}|\right)^q O_p(n^{-q/2})$$

holds and together with

$$\sum_{j=1}^n\sum_{s=1}^d|a_{t,(j-1)d+s}|=\sum_{j=1}^{dn}|a_{t,j}|\leq\left(dn\sum_{j=1}^{dn}a_{t,j}^2\right)^{1/2}\leq\sqrt{dn}\rho(\mathbf{\Gamma}_{dn}^{-1/2})=O(\sqrt{dn})\qquad\text{(A.5)}$$

by Cauchy-Schwarz, this term is of order $O_P(d^{q/2})$ and bounded in probability for fixed $d$. Now, we turn to the first term. We define $P_{j-m}\underline{X}_j=E(\underline{X}_j-\underline{\mu}|\mathcal{F}_{j-m})-E(\underline{X}_j-\underline{\mu}|\mathcal{F}_{j-m-1})$, where $\mathcal{F}_t=\sigma(\underline{X}_s-\underline{\mu},s\leq t)$ and set $M_{m,n,t}=\sum_{j=1}^n\underline{A}_t^T(j)\left\{E(\underline{X}_j-\underline{\mu}|\mathcal{F}_{j-m})-E(\underline{X}_j-\underline{\mu}|\mathcal{F}_{j-m-1})\right\}$. This yields $\sum_{j=1}^n\underline{A}_t^T(j)(\underline{X}_j-\underline{\mu})=\sum_{m=0}^{\infty}M_{m,n,t}$ almost surely and now the desired result follows from $\| \sum_{m=0}^{\infty} M_{m,n,t}\|_q^q \leq (\sum_{m=0}^{\infty} \|M_{m,n,t}\|_q)^q < \infty$ uniformly in $t$. By Proposition 4 in Dedecker and Doukhan (2003), we get $\|M_{m,n,t}\|_q \leq (2q\sum_{i=1}^n b_{i,m,n,t})^{1/2}$, where

$$b_{i,m,n,t} = \max_{i\leq l\leq n} \|\underline{A}_t^T(i)P_{i-m}\underline{X}_i \sum_{k=i}^{l} E(\underline{A}_t^T(k)P_{k-m}\underline{X}_k|\mathcal{M}_i)\|_{q/2}$$

with $\mathcal{M}_i = \sigma(\underline{A}_t^T(k)P_{k-m}\underline{X}_k, 0 \leq k \leq i)$. For the conditional expectations above, we obtain $E(\underline{A}_t^T(i)P_{i-m}\underline{X}_i|\mathcal{M}_i) = \underline{A}_t^T(i)P_{i-m}\underline{X}_i$ for $k = i$ and

$$E(\underline{A}_t^T(k)P_{k-m}\underline{X}_k|\mathcal{M}_i) = \underline{A}_t^T(k)E(E(\underline{X}_k|\mathcal{F}_{k-m})|\mathcal{M}_i) - \underline{A}_t^T(k)E(E(\underline{X}_k|\mathcal{F}_{k-m-1})|\mathcal{M}_i) = 0$$

for all $k > i$ due to $\mathcal{M}_i \subset \sigma(\mathcal{F}_{k-m-1})$. This leads to

$$b_{i,m,n,t} = \|(\underline{A}_t^T(i)P_{i-m}\underline{X}_i)^2\|_{q/2} \leq \underline{A}_t^T(i)\underline{A}_t(i)\|(P_{i-m}\underline{X}_i)^T P_{i-m}\underline{X}_i\|_{q/2}$$

by Cauchy-Schwarz, where the first factor equals $\sum_{s=(i-1)d+1}^{id} a_{ts}^2$ and the second is bounded by

$$\sum_{p=1}^{d} (E(P_{i-m}X_{p,i})^q)^{2/q} \leq d \max_{p=1,\dots,d} (E(P_{i-m}X_{p,i})^q)^{2/q} = d \max_{p=1,\dots,d} (E(P_0X_{p,m})^q)^{2/q}.$$

This results in

$$\|M_{m,n,t}\|_q^2 \leq 2q\sum_{i=1}^{n} \sum_{s=(i-1)d+1}^{id} a_{ts}^2 d \max_{p=1,\dots,d} (E(P_0X_{p,m})^q)^{1/2} = \left(2qd\sum_{i=1}^{dn} a_{ti}^2\right) \max_{p=1,\dots,d} \|P_0X_{p,m}\|_q^2 \quad \text{(A.6)}$$

and $(\sum_{m=0}^{\infty} \|M_{m,n,t}\|_q)^q = O(d^{q/2})$ by similar arguments used to get (A.5). In particular, the last term is bounded in probability for fixed $d$, which concludes this proof. □

**Lemma A.4.** *Under the assumptions of Theorem 4.1, $\frac{1}{\sqrt{n}}\sum_{t=1}^n \underline{\widetilde{Y}}_t^*$ converges in distribution in probability to a centered normal distribution with variance obtained in Lemma A.2.*

*Proof.* By Lemma A.1(ii), we may consider $\underline{b}^T\frac{1}{\sqrt{n}}\sum_{t=1}^n \underline{\widetilde{Y}}_t^{*,k}$ for some $\mathbb{R}^d$-valued vector $\underline{b}$ and prove its asymptotic normality by using the Cramér-Wold device. We have

$$\underline{b}^T\frac{1}{\sqrt{n}}\sum_{t=1}^{n} \underline{\widetilde{Y}}_t^{*,k} = \frac{1}{\sqrt{n}}\underline{b}^T\mathbf{J}(\mathbf{\Gamma}_{dn,k}^{1/2})\underline{\widetilde{Z}}^* = \sum_{i=1}^{dn} n^{-1/2}\underline{b}^T\mathbf{J}\Gamma_{dn,k}^{1/2}(\bullet,i)\widetilde{Z}_i^* = \sum_{i=1}^{dn} U_i^* \quad \text{(A.7)}$$

with an obvious notation for $U_i^*$, where $\Gamma_{dn,k}^{1/2}(\bullet,i)$ denotes the $i$th column of $\mathbf{\Gamma}_{dn,k}^{1/2}$. Then, the desired result follows from a CLT for triangular arrays [cf. Billingsley (1995, p.362)] which follows if the Lyapunov condition (for $\delta = 2$), i.e.

$$\frac{1}{(Var^*(\sum_{i=1}^{dn} U_i^*))^2}\sum_{i=1}^{dn} E^*(U_i^{*4}) \to 0 \quad \text{(A.8)}$$

in probability as $n \to \infty$ is satisfied. Considering the denominator, we get that

$$\left(\sum_{i=1}^{dn} E^*(U_i^{*2})\right)^2 = \left(\sum_{i=1}^{dn} \frac{1}{n}\underline{b}^T\mathbf{J}\Gamma_{dn,k}^{1/2}(\bullet,i)\underline{b}^T\mathbf{J}\Gamma_{dn,k}^{1/2}(\bullet,i)\right)^2 = \left(\frac{1}{n}\underline{b}^T\mathbf{J}\mathbf{\Gamma}_{dn,k}\mathbf{J}^T\underline{b}\right)^2$$

is bounded from below and from above due to $\rho(\mathbf{\Gamma}_{dn,k} - \mathbf{\Gamma}_{dn}) \to 0$ for any $k \to \infty$ and (A3). With Lemma A.3, we get for the numerator

$$\sum_{i=1}^{dn} E^*(U_i^{*4}) = \sum_{i=1}^{dn} \frac{1}{n^2}\left(\underline{b}^T\mathbf{J}\Gamma_{dn,k}^{1/2}(\bullet,i)\right)^4 E^*(\widetilde{Z}_i^{*4}) \leq \sum_{i=1}^{dn} \frac{1}{n^2}|\underline{b}|_2^4|\mathbf{J}\Gamma_{dn,k}^{1/2}(\bullet,i)|_2^4 O_P(d^2)$$

and with $|\mathbf{J}\Gamma_{dn,k}^{1/2}(\bullet,i)|_2^4 = O(k^4d^2)$ uniformly in $i$, we obtain $\sum_{i=1}^{dn} E^*(U_i^{*4}) = O_P(d^5k^4/n)$. Altogether, this leads to

$$\frac{1}{(Var^*(\sum_{j=1}^n U_j^*))^2}\sum_{j=1}^n E^*(U_j^{*4}) = O_P\left(\frac{k^4}{n}\right) = o_P(1)$$

for some appropriate sequence $k = k(n)$ that satisfies $\log^2(n)s_k = o(1)$ and $k^4 = o(n)$, which is assured to exist by (A1) with some $g > 0$. □

**Proof of Theorem 4.2.**
Let $\underline{\widetilde{J}}_n^*(\omega) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^n \underline{\widetilde{Y}}_t^* e^{-it\omega}$ and $\underline{\widetilde{J}}_n^{*,k}(\omega) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^n \underline{\widetilde{Y}}_t^{*,k} e^{-it\omega}$ be the discrete Fourier transforms based on $\underline{Y}_1^*,\ldots,\underline{Y}_n^*$ and $\underline{Y}_1^{*,k},\ldots,\underline{Y}_n^{*,k}$ as defined in (A.4) and in Lemma A.1, respectively. The corresponding periodograms are denoted by $\widetilde{\mathbf{I}}_n^*(\omega) = \underline{\widetilde{J}}_n^*(\omega)\underline{\widetilde{J}}_n^{*H}(\omega)$ and $\widetilde{\mathbf{I}}_n^{*,k}(\omega) = \underline{\widetilde{J}}_n^{*,k}(\omega)\underline{\widetilde{J}}_n^{*,kH}(\omega)$ such that

$$\widetilde{\mathbf{f}}^*(\omega) \quad = \quad \frac{1}{n}\sum_{j=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} K_b(\omega-\omega_j)\widetilde{\mathbf{I}}_n^*(\omega_j)$$

and $\widetilde{\mathbf{f}}^{*,k}(\omega)$ analogue with $\widetilde{\mathbf{I}}_n^*(\omega_j)$ replaced by $\widetilde{\mathbf{I}}_n^{*,k}(\omega_j)$ are approximations to the bootstrap kernel spectral density estimator $\widehat{\mathbf{f}}^*(\omega)$. The proof proceeds through a sequence of lemmas. Lemma A.5 gives the justification for considering

$$\sqrt{nb}(\widetilde{f}_{pq}^*(\omega) - \widetilde{f}_{pq}(\omega)) = \sqrt{nb}(\widetilde{f}_{pq}^*(\omega) - E^*(\widetilde{f}_{pq}^*(\omega))) + \sqrt{nb}(E^*(\widetilde{f}_{pq}^*(\omega)) - \widetilde{f}_{pq}(\omega))$$

in the following and to prove the CLT for these expressions, where $\widetilde{\mathbf{f}}$ as defined in Lemma A.5 below. Lemma A.6 gives the covariance structure of the stochastic leading term above, Lemma A.7 deals with the asymptotics of the bias term and Lemma A.8 provides asymptotic normality.

**Lemma A.5.** *Under the assumptions of Theorem 4.2, it holds*

$$(i) \quad \underline{J}_n^*(\omega) - \underline{\widetilde{J}}_n^*(\omega) = o_{P^*}(1) \quad \textit{and} \quad (ii) \quad \underline{\widetilde{J}}_n^*(\omega) - \underline{\widetilde{J}}_n^{*,k}(\omega) = o_{P^*}(1)$$

*uniformly in $\omega$, respectively. Further, it holds*

$$(iii) \quad \sqrt{nb}(\widehat{\mathbf{f}}^*(\omega) - \widetilde{\mathbf{f}}^*(\omega)) = o_{P^*}(1) \quad \textit{and} \quad (iv) \quad \sqrt{nb}(\widetilde{\mathbf{f}}^*(\omega) - \widetilde{\mathbf{f}}^{*,k}(\omega)) = o_{P^*}(1)$$

*and, for $\widetilde{\mathbf{f}}(\omega) = \frac{1}{2\pi}\sum_{h=-(n-1)}^{n-1}\mathbf{C}(h)e^{-ih\omega}$, we have (v) $\sqrt{nb}\left(\widetilde{\mathbf{f}}(\omega) - \check{\mathbf{f}}(\omega)\right) = o_P(1)$ for all $\omega$.*

*Proof.* (i) Let $\mathbf{J}_\omega = (e^{-i1\omega},\ldots,e^{-in\omega})\otimes\mathbf{I}_d$ and $\underline{b}\in\mathbb{C}^d$. Then, we have

$$\underline{b}^T(\underline{J}_n^*(\omega) - \underline{\widetilde{J}}_n^*(\omega)) = \frac{1}{\sqrt{2\pi n}}\underline{b}^T\mathbf{J}_\omega\left((\mathbf{\Gamma}_{dn})^{1/2}(\underline{Z}^* - \underline{\widetilde{Z}}^*) + \left((\widehat{\mathbf{\Gamma}}_{\kappa,l})^{1/2} - (\mathbf{\Gamma}_{dn})^{1/2}\right)\underline{Z}^*\right)$$

and analogue to the proof of Lemma A.1 we obtain the claimed result, where uniformity follows from the fact that $\underline{b}^T\mathbf{J}_\omega\overline{\mathbf{J}_\omega}^T\overline{\underline{b}}$ is independent of $\omega$. Part (ii) follows similarly. (iii) Plugging-in for $\widehat{\mathbf{f}}^*(\omega)$ and $\widetilde{\mathbf{f}}^*(\omega)$, yields

$$\begin{aligned}\rho(\sqrt{nb}(\widehat{\mathbf{f}}^*(\omega) - \widetilde{\mathbf{f}}^*(\omega))) \quad &\leq \quad \sqrt{\frac{b}{n}}\sum_{j=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} K_b(\omega-\omega_j)\rho\left(\underline{J}_n^*(\omega_j)\left(\underline{J}_n^*(\omega_j) - \underline{\widetilde{J}}_n^*(\omega_j)\right)^H\right) \\ &\quad + \sqrt{\frac{b}{n}}\sum_{j=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} K_b(\omega-\omega_j)\rho\left(\left(\underline{J}_n^*(\omega_j) - \underline{\widetilde{J}}_n^*(\omega_j)\right)\left(\underline{\widetilde{J}}_n^*(\omega_j)\right)^H\right) \\ &= \quad A_1 + A_2.\end{aligned}$$

By using part (i) of this lemma, we get

$$A_1 \leq \sqrt{\frac{b}{n}} \sum_{j=-\lfloor \frac{n-1}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} K_b(\omega - \omega_j) \rho\left(\underline{J}_n^*(\omega_j)\right) \rho\left(\underline{J}_n^{*H}(\omega_j) - \underline{\widetilde{J}}_n^{*H}(\omega_j)\right) = O_{P^*}\left(\sqrt{nb} \log^2(n) r_{l,n}\right)$$

and an analogue result for $A_2$. Part (iv) follows in the same way and is omitted. (v) By plugging-in for $\widetilde{\mathbf{f}}(\omega)$ and $\check{\mathbf{f}}(\omega)$, we get

$$|\sqrt{nb}(\widetilde{\mathbf{f}}(\omega) - \check{\mathbf{f}}(\omega))|_1 \leq \frac{\sqrt{nb}}{2\pi} \sum_{h=-(n-1)}^{n-1} (1 - \kappa_l(h)) |\mathbf{C}(h)|_1 + \frac{\sqrt{nb}}{2\pi} \sum_{h=-(n-1)}^{n-1} \kappa_l(h) \left|\mathbf{C}(h) - \widehat{\mathbf{C}}(h)\right|_1$$

and the first summand above is of order $O_P(\sqrt{nb} \cdot s_l)$ and the second one is $O_P(\sqrt{b}l)$. □

**Lemma A.6.** *Under the assumptions of Theorem 4.2, it holds*

$$nbCov^*\left(\widetilde{f}_{pq}^*(\omega), \widetilde{f}_{rs}^*(\lambda)\right) = \left(f_{pr}(\omega)\overline{f_{qs}(\omega)}\delta_{\omega\lambda} + f_{ps}(\omega)\overline{f_{qr}(\omega)}\tau_{0,\pi}\right) \frac{1}{2\pi} \int K^2(v)dv + o_P(1)$$

*for all* $p, q, r, s = 1, \ldots, d$ *and all* $\omega, \lambda \in [0, \pi]$.

*Proof.* We have

$$nbCov^*(\widetilde{f}_{pq}^*(\omega), \widetilde{f}_{rs}^*(\lambda)) = \frac{b}{n} \sum_{k_1,k_2=-\lfloor \frac{n-1}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} K_b(\omega - \omega_{j_1}) K_b(\lambda - \omega_{j_2}) Cov^*(\widetilde{I}_{n,pq}^*(\omega_{j_1}), \widetilde{I}_{n,rs}^*(\omega_{j_2})) \quad \text{(A.9)}$$

and the conditional covariance on the last right-hand side above becomes

$$\frac{1}{4\pi^2 n^2} \sum_{t_1,t_2,t_3,t_4=1}^{n} \sum_{i_1,i_2,i_3,i_4=1}^{dn} \Gamma_{dn}^{1/2}((t_1-1)d+p, i_1) \Gamma_{dn}^{1/2}((t_2-1)d+q, i_2) \Gamma_{dn}^{1/2}((t_3-1)d+r, i_3)$$
$$\times \Gamma_{dn}^{1/2}((t_4-1)d+s, i_4) Cov^*(\widetilde{Z}_{i_1}^* \widetilde{Z}_{i_2}^*, \widetilde{Z}_{i_3}^* \widetilde{Z}_{i_4}^*) e^{-i(t_1-t_2)\omega_{j_1}} e^{i(t_3-t_4)\omega_{j_2}} \quad \text{(A.10)}$$

and due to i.i.d. resampling, we have

$$Cov^*(\widetilde{Z}_{i_1}^* \widetilde{Z}_{i_2}^*, \widetilde{Z}_{i_3}^* \widetilde{Z}_{i_4}^*) = \begin{cases} 1, & i_1 = i_3, i_2 = i_4 \text{ or } i_1 = i_4, i_2 = i_3 \\ E^*(\widetilde{Z}_i^{*4}) - 3, & i_1 = i_2 = i_3 = i_4 \\ 0, & \text{otherwise} \end{cases}. \quad \text{(A.11)}$$

Both combinations of the first case in (A.11) together with (A.10) lead to

$$\left(\frac{1}{2\pi n} \sum_{t_1,t_3=1}^{n} C_{pr}(t_1 - t_3) e^{-it_1\omega_{j_1}} e^{it_3\omega_{j_2}}\right) \left(\frac{1}{2\pi n} \sum_{t_2,t_4=1}^{n} C_{qs}(t_2 - t_4) e^{it_2\omega_{j_1}} e^{-it_4\omega_{j_2}}\right)$$
$$+ \left(\frac{1}{2\pi n} \sum_{t_1,t_4=1}^{n} C_{ps}(t_1 - t_4) e^{-it_1\omega_{j_1}} e^{-it_4\omega_{j_2}}\right) \left(\frac{1}{2\pi n} \sum_{t_2,t_3=1}^{n} C_{qr}(t_2 - t_3) e^{it_2\omega_{j_1}} e^{it_3\omega_{j_2}}\right)$$
$$= f_{pr}(\omega_{j_1})\overline{f_{qs}(\omega_{j_1})} \mathbf{1}(j_1 = j_2) + f_{ps}(\omega_{j_1})\overline{f_{qr}(\omega_{j_1})} \mathbf{1}(j_1 = -j_2) + o(1) \quad \text{(A.12)}$$

uniformly in $\omega_{j_1}$ and $\omega_{j_2}$, respectively. To handle the second case in (A.11), observe that we may replace all entries of $\mathbf{\Gamma}_{dn}^{1/2}$ by the corresponding entries of its $k$-banded version $\mathbf{\Gamma}_{dn,k}^{1/2}$ in (A.10) by Lemma A.5. Therefore, we obtain

$$\frac{1}{4\pi^2 n^2}\sum_{i=1}^{dn}\left(\sum_{t_1=1}^{n}\Gamma_{dn,k}^{1/2}((t_1-1)d+p,i)e^{-it_1\omega_{j_1}}\right)\left(\sum_{t_2=1}^{n}\Gamma_{dn,k}^{1/2}((t_2-1)d+q,i)e^{it_2\omega_{j_1}}\right)$$
$$\times\left(\sum_{t_3=1}^{n}\Gamma_{dn,k}^{1/2}((t_3-1)d+r,i)e^{it_3\omega_{j_2}}\right)\left(\sum_{t_4=1}^{n}\Gamma_{dn,k}^{1/2}((t_4-1)d+s,i)e^{-it_4\omega_{j_2}}\right)\left(E^*(\widetilde{Z}_1^{*4})-3\right)$$
$$= O_P\left(\frac{k^4}{n}\right)$$

uniformly in $\omega_{j_1}$ and $\omega_{j_2}$ due to the banded shape of $\mathbf{\Gamma}_{dn,k}^{1/2}$ and Lemma A.3. Together with (A.9), the second case in (A.11) becomes $O_P(bk^4) = o_P(1)$ under the assumptions. □

**Lemma A.7.** *Under the assumptions of Theorem 4.2, it holds*

$$E^*\left(\widetilde{f}_{pq}^*(\omega)\right)-\widetilde{f}_{pq}(\omega) = b^2 f_{pq}''(\omega)\frac{1}{4\pi}\int_{-\pi}^{\pi}v^2K(v)dv + o_P(b^2)$$

*for all* $p,q=1,\ldots,d$ *and all* $\omega$.

*Proof.* Straightforward calculations yield

$$E^*\left(\widetilde{\mathbf{f}}^*(\omega)\right)-\widetilde{\mathbf{f}}(\omega) = \frac{1}{n}\sum_{j=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}K_b(\omega-\omega_j)\frac{1}{2\pi}\sum_{h=-(n-1)}^{n-1}\left(1-\frac{|h|}{n}\right)\mathbf{C}(h)\left(e^{-ih\omega_j}-e^{-ih\omega}\right)$$
$$+\left(\frac{1}{n}\sum_{j=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}K_b(\omega-\omega_j)-1\right)\frac{1}{2\pi}\sum_{h=-(n-1)}^{n-1}\mathbf{C}(h)e^{-ih\omega},$$

where the second term is negligible thanks to $|\frac{1}{n}\sum_{j=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}K_b(\omega-\omega_j)-1| = O(\frac{1}{nb})$. The first term can be treated as in Lemma 7.4 in Jentsch and Kreiss (2010), which concludes this proof. □

**Lemma A.8.** *Under the assumptions of Theorem 4.2,* $\sqrt{nb}(\widetilde{f}_{pq}^*(\omega_l)-E^*(\widetilde{f}_{pq}^*(\omega_l)) : p,q = 1,\ldots,d; l=1,\ldots,s)$ *converges in distribution in probability to a centered* $d^2s$*-dimensional normal distribution with covariance matrix as obtained in Lemma A.6.*

*Proof.* We prove asymptotic normality only for $\sqrt{nb}(\widetilde{f}_{pq}^*(\omega)-E^*(\widetilde{f}_{pq}^*(\omega)))$ in the following and a more general result follows from the Cramér-Wold device. First note that the quantity of interest may be expressed as a generalized quadratic form, i.e.

$$\sqrt{nb}\left(\widetilde{f}_{pq}^*(\omega)-E^*\left(\widetilde{f}_{pq}^*(\omega)\right)\right)$$
$$= \sum_{i_1,i_2=1}^{dn}\frac{\sqrt{b}}{2\pi\sqrt{n}}\sum_{t_1,t_2=1}^{n}\left(\frac{1}{n}\sum_{j=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}K_b(\omega-\omega_j)e^{-i(t_1-t_2)\omega_j}\right)$$
$$\times\Gamma_{dn}^{1/2}((t_1-1)d+p,i_1)\Gamma_{dn}^{1/2}((t_2-1)d+q,i_2)\left(\widetilde{Z}_{i_1}^*\widetilde{Z}_{i_2}^*-E^*(\widetilde{Z}_{i_1}^*\widetilde{Z}_{i_2}^*)\right)$$
$$= \sum_{i_1i_2=1}^{dn}w_{i_1i_2}(\widetilde{Z}_{i_1}^*,\widetilde{Z}_{i_2}^*)$$
$$= \sum_{1\leq i_1<i_2\leq dn}W_{i_1i_2}+\sum_{i=1}^{dn}w_{ii}(\widetilde{Z}_i^*,\widetilde{Z}_i^*)$$

with an obvious notation for $w_{i_1 i_2}(\widetilde{Z}^*_{i_1}, \widetilde{Z}^*_{i_2})$ and with $W_{i_1 i_2} := w_{i_1 i_2}(\widetilde{Z}^*_{i_1}, \widetilde{Z}^*_{i_2}) + w_{i_2 i_1}(\widetilde{Z}^*_{i_2}, \widetilde{Z}^*_{i_1})$. Due to i.i.d. resampling, we can apply Theorem 2.1 of deJong (1987) to the quadratic form. It is clean by an easy computation and its variance is bounded, such that it remains to show that

$$(i) \max_{1\leq i_1\leq dn} \sum_{i_2=1}^{dn} E^*(|W_{i_1 i_2}|^2) \to 0 \quad \text{and} \quad (ii) \frac{E^*|\sum_{1\leq i_1<i_2\leq dn} W_{i_1 i_2}|^4}{(E^*|\sum_{1\leq i_1<i_2\leq dn} W_{i_1 i_2}|^2)^2} \to 3$$

hold in probability, respectively. To show (i), examplarily, we consider only the first term of $|W_{i_1 i_2}|^2$ in more detail, that is, $|w_{i_1 i_2}(\widetilde{Z}^*_{i_1}, \widetilde{Z}^*_{i_2})|^2$ and we get

$$\begin{aligned}
& E^*(|w_{i_1 i_2}(\widetilde{Z}^*_{i_1}, \widetilde{Z}^*_{i_2})|^2) \qquad\qquad (A.13)\\
= & \frac{b}{4\pi^2 n} \sum_{t_1,t_2,t_3,t_4=1}^{n} \left(\frac{1}{n} \sum_{j_1=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} K_b(\omega-\omega_{j_1}) e^{-i(t_1-t_2)\omega_{j_1}}\right)\left(\frac{1}{n} \sum_{j_2=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} K_b(\omega-\omega_{k_2}) e^{i(t_3-t_4)\omega_{j_2}}\right)\\
& \times\Gamma^{1/2}_{dn}((t_1-1)d+p,i_1)\Gamma^{1/2}_{dn}((t_2-1)d+q,i_2)\Gamma^{1/2}_{dn}((t_3-1)d+p,i_1)\Gamma^{1/2}_{dn}((t_4-1)d+q,i_2)Var^*(\widetilde{Z}^*_{i_1}\widetilde{Z}^*_{i_2}).
\end{aligned}$$

The conditional variance above equals $1+(E^*(\widetilde{Z}^{*4}_1)-2)1(i_1=i_2)$ and we may replace all entries of $\mathbf{\Gamma}^{1/2}_{dn}$ by the corresponding entries of its $k$-banded version $\mathbf{\Gamma}^{1/2}_{dn,k}$ by Lemma A.5 in the following. For any fixed $i_1$, we obtain by summing over $i_2$ for the first case

$$\begin{aligned}
& \frac{b}{4\pi^2 n} \sum_{t_1,t_2,t_3,t_4=1}^{n} \left(\frac{1}{n} \sum_{j_1=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} K_b(\omega-\omega_{j_1}) e^{-i(t_1-t_2)\omega_{j_1}}\right)\left(\frac{1}{n} \sum_{j_2=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} K_b(\omega-\omega_{j_2}) e^{i(t_3-t_4)\omega_{j_2}}\right)\\
& \times\Gamma^{1/2}_{dn,k}((t_1-1)d+p,i_1)\Gamma_{dn,k}((t_2-1)d+q,(t_4-1)d+q)\Gamma^{1/2}_{dn,k}((t_3-1)d+p,i_1)\\
= & O\left(\frac{k^2}{n}\right)+o\left(bk^2\right)
\end{aligned}$$

and with use of Lemma A.3, we get for the second case

$$\begin{aligned}
& \frac{b}{4\pi^2 n} \sum_{t_1,t_2,t_3,t_4=1}^{n} \left(\frac{1}{n} \sum_{j_1=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} K_b(\omega-\omega_{j_1}) e^{-i(t_1-t_2)\omega_{j_1}}\right)\left(\frac{1}{n} \sum_{j_2=-\lfloor\frac{n-1}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor} K_b(\omega-\omega_{j_2}) e^{i(t_3-t_4)\omega_{j_2}}\right)\\
& \times\Gamma^{1/2}_{dn,k}((t_1-1)d+p,i_1)\Gamma^{1/2}_{dn,k}((t_2-1)d+q,i_2)\Gamma^{1/2}_{dn,k}((t_3-1)d+p,i_1)\Gamma^{1/2}_{dn,k}((t_4-1)d+q,i_2)\\
= & O_P\left(\frac{bk^4}{n}\right)
\end{aligned}$$

uniformly in $i_1$, respectively, and both terms above vanish asymptotically for some suitably chosen sequence $k$ which is possible by the imposed conditions. Being concerned with (ii), we consider first the numerator and get

$$E^*\left(\left|\sum_{1\leq i_1<i_2\leq dn} W_{i_1 i_2}\right|^4\right) = E^*\left(\left|\sum_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{dn} w_{i_1 i_2}(\widetilde{Z}^*_{i_1}, \widetilde{Z}^*_{i_2})\right|^4\right)$$

and by expanding the above expression, we see that the most contributing case is where we have four twins of equal indices. All other cases are of lower order. If we take all summands above into account, only three combinations of twins do not vanish and each of them yields $const^2 \cdot f^4_{pq}(\omega)$. For the denominator, we get $(const \cdot f^2_{pq}(\omega))^2$ which concludes this proof. For details compare for instance the proof of Theorem 2 in Jentsch (2012). □

**Proof of Theorem 5.1.**
Under assumptions (A1′) and (A2′), we get the same bounds as obtained in Theorems 2.1, and respectively, and $|\mathbf{C}(h)|_1 \leq d^2 \sup_{n\in\mathbb{N}} \sup_{i,j=1,\ldots,d(n)} |C_{ij}(h)|$ leads to the first part of (i). By similar arguments as employed in the proof of Corollary 2.1, we get also the second part under (A3′) and convergence to zero in probability under the imposed conditions. Analogue to the proof of Corollary 2.1, we get (ii) and (iii) by exploiting the bound in (A.3) for the Cholesky factorization. □

**Proof of Theorem 5.2.**
We follow the proof of Theorem 4.1 and adopt the notation therein. First, analogue to the proof of Lemma A.1, we get for any real-valued sequence $\underline{b} = \underline{b}(d(n))$ of $d(n)$-dimensional vectors with

$$0 < M_1 \leq |\underline{b}(d(n))|_2^2 \leq M_2 < \infty \quad \text{for all} \quad n \in \mathbb{N} \tag{A.14}$$

that the following holds

$$(i) \quad \underline{b}^T \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \underline{Y}_t^* \right) = \underline{b}^T \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \underline{\widetilde{Y}}_t^* \right) + O_{P^*}(\log^2(dn) d^2 \widetilde{r}_{l,n}),$$

$$(ii) \quad \underline{b}^T \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \underline{\widetilde{Y}}_t^* \right) = \underline{b}^T \left( \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \underline{\widetilde{Y}}_t^{*,k} \right) + O_{P^*}(\log^2(dn) d^2 \widetilde{s}_k),$$

where $\widetilde{r}_{l,n}$ is defined in Theorem 5.1 and $\widetilde{s}_k = \sum_{h=k+1}^{\infty} \{\sup_{n\in\mathbb{N}} \sup_{i,j=1,\ldots,d(n)} |C_{ij}(h)|\}$. Both $O_{P^*}$-terms above vanish under the imposed conditions. Similar to the proof of Lemma A.4, we want to show asymptotic normality for (A.7) and, therefore, we have to check the Lyapunov condition (A.8). For the denominator, we get that

$$\left( \sum_{i=1}^{dn} E^*(U_i^{*2}) \right)^2 = \left( \sum_{i=1}^{dn} \frac{1}{n} \underline{b}^T \mathbf{J} \Gamma_{dn,k}^{1/2}(\bullet, i) \underline{b}^T \mathbf{J} \Gamma_{dn,k}^{1/2}(\bullet, i) \right)^2 = \left( \frac{1}{n} \underline{b}^T \mathbf{J} \mathbf{\Gamma}_{dn,k} \mathbf{J}^T \underline{b} \right)^2$$

is bounded from below and from above due to (A.14), (A3′) and $\rho(\mathbf{\Gamma}_{dn,k} - \mathbf{\Gamma}_{dn}) \to 0$ for a suitable sequence $k \to \infty$ assured to exist by the imposed assumptions. For the numerator, we get

$$\sum_{i=1}^{dn} E^*(U_i^{*4}) \leq \sum_{i=1}^{dn} \frac{1}{n^2} |\underline{b}|_2^4 |\mathbf{J} \Gamma_{dn,k}^{1/2}(\bullet, i)|_2^4 O_P(d^2) = O_P(d^5 k^4 / n),$$

where Lemma A.3 for increasing time series dimension has been used. This gives

$$\frac{1}{(Var^*(\sum_{j=1}^{n} U_j^*))^2} \sum_{j=1}^{n} E^*(U_j^{*4}) = O_P\left( \frac{d^5 k^4}{n} \right) = o_P(1)$$

for some appropriate sequence $k = k(n)$ that satisfies $\log^2(dn) d^2 \widetilde{s}_k = o(1)$ and $d^5 k^4 = o(n)$, which is assured to exist by (A1′) with some sufficiently large $g > 0$. Further, we get

$$\begin{aligned} |v^2 - \widehat{v}^2| &= \frac{1}{n} \underline{b}^T \mathbf{J} \left\{ E(\underline{XX}^T) - E^*(\underline{XX}^T) \right\} \mathbf{J}^T \underline{b} = \frac{1}{n} \underline{b}^T \mathbf{J} \{ \mathbf{\Gamma}_{dn} - \widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon} \} \mathbf{J}^T \underline{b} \\ &\leq \left( \frac{1}{n} \underline{b}^T \mathbf{J} \mathbf{J}^T \underline{b} \right)^{1/2} \left( \frac{1}{n} \underline{b}^T \mathbf{J} \{ \mathbf{\Gamma}_{dn} - \widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon} \} \{ \mathbf{\Gamma}_{dn} - \widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon} \}^T \mathbf{J}^T \underline{b} \right)^{1/2} \\ &\leq |\underline{b}|_2^2 \rho(\mathbf{\Gamma}_{dn} - \widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon}) \end{aligned}$$

which concludes the proof as $\rho(\mathbf{\Gamma}_{dn} - \widehat{\mathbf{\Gamma}}_{\kappa,l}^{\epsilon}) = o_P(1)$ and $|\underline{b}_2^2| = O(1)$ by assumption. □

## References

[1] Billingsley, P. (1999) *Convergence of Probability Measures, 2nd ed.* Chichester, Wiley.

[2] Dedecker, J. and Doukhan, P. (2003). A new covariance inequality and applications. *Stochastic Process. Appl.* **106** 63–80.

[3] De Jong, P. (1987). A central limit theorem for generalized quadratic forms. *Probab. Theory Relat. Fields* **75** 261–277.

[4] Drmac, Z., Omladic, M. and Veselic, K. (1994). On the perturbation of the cholesky factorization. *SIAM Journal on Matrix Analysis and Applications* **15** 1319–1332.

[5] Horn, R.A. and Johnson, C.R. (1990). *Matrix analysis.* Cambridge University Press, New York.

[6] Jentsch, C. (2012). A new frequency domain approach of testing for covariance stationarity and for periodic stationarity in multivariate linear processes. *J. Time Ser. Anal.* **33** 177–192.

[7] Jentsch, C. and Kreiss, J.-P. (2010). The multiple Hybrid Bootstrap - Resampling multivariate linear processes. **J. Multivariate Anal. 101** 2320-2345.

[8] Jentsch, C. and Politis, D.N. (2014). Covariance matrix estimation and linear process bootstrap for multivariate time series of possibly increasing dimension. Submitted.

[9] McMurry, T.L. and Politis, D.N. (2010). Banded and tapered estimates for autocovariance matrices and the linear process bootstrap. *J. Time Ser. Anal.* **31** 471–482. CORRIGENDUM: J. Time Ser. Anal. **33**, 2012.

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