

# A NEW FREQUENCY DOMAIN APPROACH OF TESTING FOR COVARIANCE STATIONARITY AND FOR PERIODIC STATIONARITY IN MULTIVARIATE LINEAR PROCESSES

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ABSTRACT. In modeling seasonal time series data, periodically (non-)stationary processes have become quite popular over the last years and it is well known that these models may be represented as higher-dimensional stationary models. In this paper, it is shown that the spectral density matrix of this higher-dimensional process exhibits a certain structure if and only if the observed process is covariance stationary. By exploiting this relationship, a new  $L_2$ -type test statistic is proposed for testing whether a multivariate periodically stationary linear process is even covariance stationary. Moreover, it is shown that this test may also be used to test for periodic stationarity. The asymptotic normal distribution of the test statistic under the null is derived and the test is shown to have an omnibus property. The paper concludes with a simulation study, where the small sample performance of the test procedure is improved by using a suitable bootstrap scheme.

## 1. INTRODUCTION

In modelling time series the challenge is to find parsimonious models that satisfactorily capture the possibly complicated dependence structure of the observed data. The aim of parsimony becomes even more important for multivariate time series, where the number of involved parameters that may have to be estimated increases dramatically. Usually, second-order stationarity (covariance stationarity) is assumed to ensure sufficient mathematical tractability of the chosen model and to make things manageable. A common procedure in modelling non-stationary time series is to standardize or to filter the observed series and then fit an appropriate stationary stochastic model to the reduced series.

However, in many situations, the assumption of stationarity is not fulfilled and there is no transformation that may be applied to the data to achieve second-order stationarity. Sometimes and particularly for seasonal time series, this is because the covariance structure of a time series possibly depends on the season, that is, the autocovariance function is periodic for all lags. For instance, time series of this kind appear in hydrology, climatology, meteorology and other geophysical sciences, but also in economics, where the observed time series are characterized by periodic variations in both the mean and covariance structure.

For time series with periodic correlation structure [cf. Hurd and Miamee (2007) for an overview], Gladyshev (1961) introduced the notion of periodically stationary processes. Further pioneering work has been done by Jones and Brelsford (1967), Pagano (1978) and Troutman (1979), who have examined fundamental properties of univariate periodic autoregressive (PAR) processes. Later periodic moving-average processes (PMA) [see Cipra (1985), Bentarzi and Hallin (1994, 1998)] and the more general class of periodic autoregressive moving-average time series

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*Date:* September 17, 2011.

*2000 Mathematics Subject Classification.* 62M10; 62G10.

*Key words and phrases.* hypothesis testing, testing for stationarity, testing for periodic stationarity, periodic time series, multivariate time series, linear process, spectral density matrix, kernel spectral density estimates, hybrid bootstrap.

(PARMA) have been considered [cf. Vecchia (1985a,b), Lund and Basawa (2000), Basawa and Lund (2001) among others]. These models are extensions of the usual ARMA models where the coefficients and the variances of the white noise process are allowed to depend on the season. Multivariate generalizations of these models have been investigated by Ula (1990, 1993), Franses and Paap (2004) and Lütkepohl (2005), but basic research still has to be done.

Time series analysis of data sequences usually involves three main steps: model identification, parameter estimation and diagnostic checking. Concerned with model identification in seasonal time series, it seems natural to decide first whether there are actually periodicities present in the data and if so to determine the period, that is, the smallest integer  $s$ , so that all autocovariances are periodic with  $s$  periods, and to choose the orders of a potential PARMA( $p,q$ ) model afterwards.

Because a periodic autocorrelation structure complicates all three steps of model building extensively and the number of parameters in a nonstationary periodic model involves a possible  $s$  fold increase in the number of parameters over that in a stationary nonperiodic model, care must be exercised in its application. For instance, Lund, Shao and Basawa (2006) have investigated parsimonious representations in the class of periodic time series models, in order to reduce the number of parameters.

The consequences of fitting a stationary AR model to data generated by a nonstationary periodic AR model have been emphasized in Tiao and Grupe (1980). In this case the resulting model is misspecified which potentially causes a considerable forecasting error as shown in their paper. As illustrated in Figure 1, it is usually not possible to distinguish between stationary and periodically stationary processes or to guess the number of periods from the plot of the data. Therefore, it is of large interest to establish powerful tools to decide whether the assumptions of covariance stationarity may be valid for a particular (multiple) seasonal time series or to determine its actual period, respectively.

Cipra (1985) developed a Durbin-type estimation procedure for the coefficients of PMA models and suggested a test for periodic structure based on corresponding asymptotic normality results. Vecchia and Ballerini (1991) proposed a test for deciding whether periodicities exist in the autocorrelation function of a seasonal time series under the assumptions of a causal periodic linear model. Their approach is based on a Fourier-transformed version of the estimated periodic autocorrelation function. In McLeod (1993, 1994) portmanteau type test statistics based on residuals of a fitted AR(I)MA model have been used and the three usual stages of model building methodology have been illustrated in detail. The graphical approach of Hurd and Gerr (1991) is based on the spectral representation for harmonizable second-order sequences and they suggest to use Goodman's coherence statistic to test for periodic correlation. Lenart, Leśkow and Synowiecki (2008) proposed a test statistic that exploits properties of the Fourier coefficients of the time-dependent autocovariance function for univariate periodically correlated time series under mixing assumptions using subsampling. Recently, Ursu and Duchesne (2009) considered vector-valued periodic autoregressive models (PVAR) and developed multivariate generalizations of theorems concerned with portmanteau-type tests obtained by McLeod (1994).

So far, most of the present literature on periodically stationary models concentrates on univariate periodic time series and/or finite parametric models as PAR and PARMA. However, multivariate models are expected to be extremely useful in practice and powerful tools for detecting periodicities in general linear models are of considerable interest.

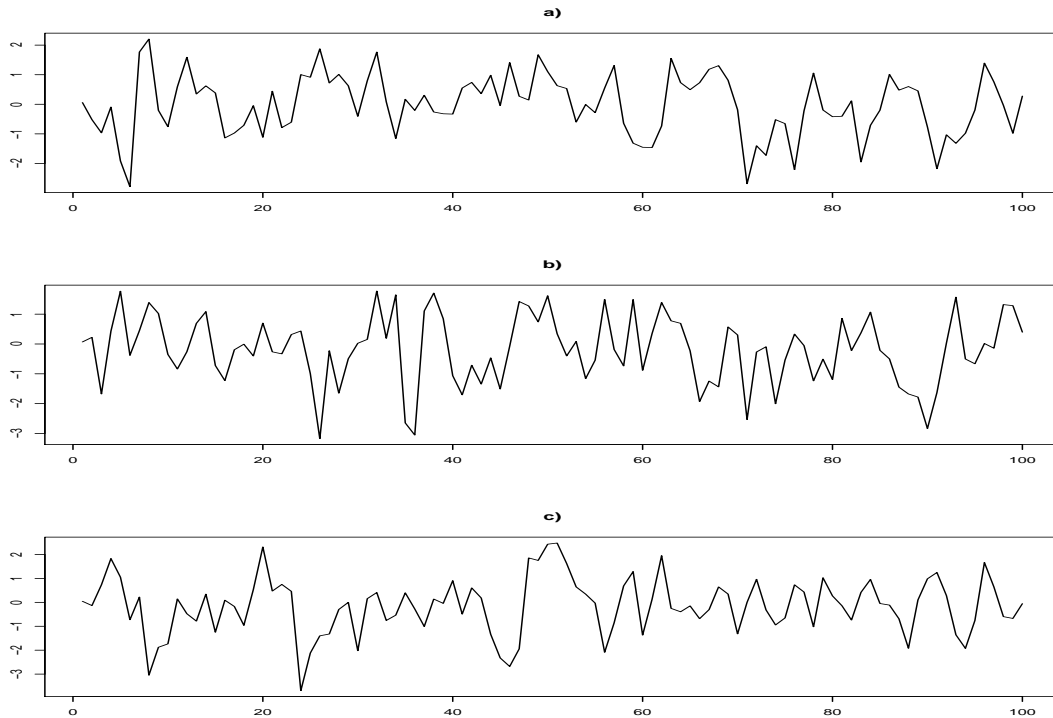


FIGURE 1. Realizations of length  $N = 100$  of a) MA(1) process with  $b_1 = 0.5$  and i.i.d  $\mathcal{N}(0,1)$  white noise, b) periodic MA(1) process with  $s = 2$  periods,  $b_1^{(1)} = 0.3$ ,  $b_1^{(2)} = 0.7$  and i.i.d  $\mathcal{N}(0,1)$  white noise, c) periodic MA(1) process with  $s = 2$  periods,  $b_1^{(1)} = b_1^{(2)} = 0.5$  and independent  $\mathcal{N}(0, \sigma_j^2)$  white noise with  $\sigma_1 = 0.8$  and  $\sigma_2 = 1.2$ .

Hence, this paper deals with the very general class of vector-valued periodically stationary linear time series (PVL) defined in (2.1) below. Observe that PVL models include the important subclasses of multivariate PAR and multivariate PARMA models.

It is well known that any  $d$ -dimensional periodically stationary process with  $s$  periods may be expressed as an  $sd$ -dimensional stationary process as shown in (2.3) for linear processes. Usually, this technique is associated with Gladyshev [cf. Gladyshev (1961)]. For instance, Pagano (1978) among others considered the relation of periodic and multiple autoregression in the univariate situation. This relationship allows using the classical theory of stationary time series as, for instance, nonparametric estimation in the frequency domain. Many relevant hypothesis about second-order properties of multivariate stationary time series may be expressed in terms of the spectral density matrix and the formulation of hypothesis in the frequency domain enjoys the advantage of a general nonparametric framework based on, for example, kernel spectral density estimators. In the context of periodically stationary processes, properties of the spectral density matrix have been considered by Troutman (1979) and Sakai (1991).

The main purpose of this paper is to present a test procedure for deciding whether the underlying PVL process with a (predetermined) number of  $s$  periods actually is covariance stationary. But, additionally, it is shown that this test may be applied also for testing whether the underlying

PVL time series is periodically stationary with some period smaller than  $s$ , which to the author's knowledge has not been investigated yet thoroughly in statistical literature. An  $L_2$ -type test statistic that estimates the integrated deviation from the null hypothesis is suggested that exploits the specific shape of a slightly adjusted spectral density matrix of the corresponding higher-dimensional process under the null hypothesis.

A CLT for the test statistic is proved and the test is shown to be an omnibus test that has power against any alternative. The finite sample performance of this test is checked in a simulation study using critical values obtained from the CLT and from an appropriate bootstrap procedure. Here, we make use of the multiple hybrid bootstrap proposed by Jentsch and Kreiss (2010), which is well suited for kernel spectral density estimation in the situation of multivariate linear processes. The use of bootstrap methods for testing periodicities in (autoregressive) time series models was already recommended by Herwartz (1998).

The paper is organized as follows. In Section 2, some preliminary results concerning  $d$ -variate periodically stationary linear processes are presented and some examples are discussed. In particular, Theorem 2.1 provides the specific structure of autocovariance function and (modified) spectral density of the corresponding  $sd$ -variate process under the null hypothesis. Section 3 deals with the construction of the test statistic, its asymptotic normality in Theorem 3.1 and its omnibus property in Theorem 3.2. A small simulation study is presented in Section 4. Finally, proofs of the main results can be found in Section 5.

## 2. PRELIMINARY RESULTS

Let  $(\underline{Y}_t, t \in \mathbb{Z})$  be an  $\mathbb{R}^d$ -valued periodically stationary linear process with  $s$  periods,  $s \in \mathbb{N}$ , that is,

$$\underline{Y}_{j+sT} = \sum_{k=-\infty}^{\infty} \mathbf{b}_k^{(j)} \underline{\epsilon}_{j+sT-k}, \quad j = 1, \dots, s, \quad T \in \mathbb{Z}, \quad (2.1)$$

where  $\mathbf{b}_k^{(j)}$ ,  $k \in \mathbb{Z}$  are  $(d \times d)$  coefficient matrices and  $\mathbf{b}_0^{(j)} = \mathbf{I}_d$  are  $(d \times d)$  unit matrices for  $j = 1, \dots, s$ . Moreover,  $(\underline{\epsilon}_t, t \in \mathbb{Z})$  are independent and centered  $d$ -variate random variables with covariance matrices  $E(\underline{\epsilon}_{j+sT} \underline{\epsilon}_{j+sT}^T) = \boldsymbol{\Sigma}_j$  for  $j = 1, \dots, s$  and  $T \in \mathbb{Z}$ , such that  $(\underline{\epsilon}_{j+sT}, T \in \mathbb{Z})$  are i.i.d. random variables for all  $j = 1, \dots, s$ .

Note that the process in (2.1) is supposed to have zero mean, that is  $E(\underline{Y}_t) = 0$ . Therefore, in practice, it is assumed that the analysis of the mean has been done, that is, the possibly periodic mean is zero or has been removed, and we are concerned here only with methods for determining whether the autocovariances have periodic structure.

In general, a periodically stationary process is *not* covariance stationary, because its  $(d \times d)$  autocovariance function is periodic with period  $s$  for all lags  $h \in \mathbb{Z}$ . More precisely, using the notation

$$\boldsymbol{\Gamma}_Y(h, m) = Cov(\underline{Y}_{m+sT}, \underline{Y}_{m+sT-h}) = E(\underline{Y}_{m+sT} \underline{Y}_{m+sT-h}^T), \quad (2.2)$$

it holds  $\boldsymbol{\Gamma}_Y(h, m) = \boldsymbol{\Gamma}_Y(h, m + sk)$  and  $\boldsymbol{\Gamma}_Y(-h, m) = \boldsymbol{\Gamma}_Y^T(h, m + h)$  for all  $m = 1, \dots, s$ ,  $k \in \mathbb{Z}$  and all lags  $h \in \mathbb{Z}$ .

In the theory of periodically stationary time series, it is a common technique to interpret them as higher-dimensional stationary processes. Precisely, the  $d$ -variate process  $(\underline{Y}_t, t \in \mathbb{Z})$  defined

in (2.1) may be represented as an  $sd$ -dimensional covariance stationary process  $(\underline{X}_t, t \in \mathbb{Z})$  according to

$$\begin{aligned} \underline{X}_T &= \begin{pmatrix} \underline{Y}_{1+sT} \\ \underline{Y}_{2+sT} \\ \vdots \\ \underline{Y}_{s+sT} \end{pmatrix} = \sum_{k=-\infty}^{\infty} \begin{pmatrix} \mathbf{b}_{sk}^{(1)} & \mathbf{b}_{sk-1}^{(1)} & \cdots & \mathbf{b}_{sk-(s-1)}^{(1)} \\ \mathbf{b}_{sk+1}^{(2)} & \mathbf{b}_{sk}^{(2)} & & \mathbf{b}_{sk-(s-2)}^{(2)} \\ \vdots & & \ddots & \vdots \\ \mathbf{b}_{sk+(s-1)}^{(s)} & \mathbf{b}_{sk+(s-2)}^{(s)} & \cdots & \mathbf{b}_{sk}^{(s)} \end{pmatrix} \begin{pmatrix} \underline{\epsilon}_{1+s(T-k)} \\ \underline{\epsilon}_{2+s(T-k)} \\ \vdots \\ \underline{\epsilon}_{s+s(T-k)} \end{pmatrix} \\ &= \sum_{k=-\infty}^{\infty} \mathbf{B}_k \underline{\epsilon}_{T-k}, \quad T \in \mathbb{Z} \end{aligned} \quad (2.3)$$

with an obvious notation for the  $(sd \times sd)$  matrices  $\mathbf{B}_k$ ,  $k \in \mathbb{Z}$  and the  $sd$ -dimensional i.i.d. white noise process  $(\underline{\epsilon}_t, t \in \mathbb{Z})$ . Further, it holds  $E(\underline{\epsilon}_t) = \mathbf{0}$  and  $E(\underline{\epsilon}_t \underline{\epsilon}_t^T) = \mathbf{\Sigma}_e$  with block-diagonal  $(sd \times sd)$  covariance matrix  $\mathbf{\Sigma}_e = \text{diag}(\mathbf{\Sigma}_1, \mathbf{\Sigma}_2, \dots, \mathbf{\Sigma}_s)$ .

Observe that due to the algebraic equivalence between multivariate stationarity of an  $sd$ -variate process and periodic stationarity with  $s$  periods of a  $d$ -variate process, the process  $(\underline{X}_t, t \in \mathbb{Z})$  introduced in (2.3) is stationary if and only if  $(\underline{Y}_t, t \in \mathbb{Z})$  defined in (2.1) is periodically stationary [cf. Gladyshev (1961), Ula (1990) and Ursu and Duchesne (2009)]. Moreover, note that covariance stationarity always implies periodic stationarity with any number of periods  $s \geq 2$  and periodic stationarity with  $s$  periods implies periodic stationarity with  $ks$  periods for any  $k \in \mathbb{N}$ .

Under sufficient summability assumptions on  $(\mathbf{B}_k, k \in \mathbb{Z})$ , the process  $(\underline{X}_t, t \in \mathbb{Z})$  is covariance stationary. Therefore, its  $(sd \times sd)$  autocovariance function  $\mathbf{\Gamma}(h)$ ,

$$\mathbf{\Gamma}(h) = \sum_{k=-\infty}^{\infty} \mathbf{B}_{k+h} \mathbf{\Sigma}_e \mathbf{B}_k^T = \begin{pmatrix} \mathbf{\Gamma}_{11}(h) & \mathbf{\Gamma}_{12}(h) & \cdots & \mathbf{\Gamma}_{1s}(h) \\ \mathbf{\Gamma}_{21}(h) & \mathbf{\Gamma}_{22}(h) & & \mathbf{\Gamma}_{2s}(h) \\ \vdots & & \ddots & \vdots \\ \mathbf{\Gamma}_{s1}(h) & \mathbf{\Gamma}_{s2}(h) & \cdots & \mathbf{\Gamma}_{ss}(h) \end{pmatrix}, \quad h \in \mathbb{Z}, \quad (2.4)$$

exists, where  $\mathbf{\Gamma}_{mn}(h)$  are  $(d \times d)$  block matrices with  $\mathbf{\Gamma}_{mn}(-h) = \mathbf{\Gamma}_{mn}^T(h)$  for all  $m, n$ . Also, under suitable assumptions,  $(\underline{X}_t, t \in \mathbb{Z})$  exhibits an  $(sd \times sd)$  spectral density matrix  $\mathbf{f}(\omega)$ , which has the representation

$$\begin{aligned} \mathbf{f}(\omega) &= \frac{1}{2\pi} \left( \sum_{k=-\infty}^{\infty} \mathbf{B}_k e^{-ik\omega} \right) \mathbf{\Sigma}_e \left( \sum_{k=-\infty}^{\infty} \mathbf{B}_k e^{-ik\omega} \right)^H \\ &= \begin{pmatrix} \mathbf{F}_{11}(\omega) & \mathbf{F}_{12}(\omega) & \cdots & \mathbf{F}_{1s}(\omega) \\ \mathbf{F}_{21}(\omega) & \mathbf{F}_{22}(\omega) & & \mathbf{F}_{2s}(\omega) \\ \vdots & & \ddots & \vdots \\ \mathbf{F}_{s1}(\omega) & \mathbf{F}_{s2}(\omega) & \cdots & \mathbf{F}_{ss}(\omega) \end{pmatrix}, \quad \omega \in [-\pi, \pi], \end{aligned} \quad (2.5)$$

where  $\mathbf{F}_{mn}(\omega)$  are  $(d \times d)$  block matrices with  $\mathbf{F}_{mn}(\omega) = \mathbf{F}_{nm}^H(\omega)$  for all  $m, n$ . Here and throughout the paper, all matrix-valued quantities are written as bold letters, all vector-valued quantities are underlined and  $(\mathbf{A}^H) \mathbf{A}^T$  indicates the (conjugate) transpose of a matrix  $\mathbf{A}$ . Moreover, minuscules  $\mathbf{f}(\omega)$  are  $(sd \times sd)$  and capitals  $\mathbf{F}_{mn}(\omega)$  are  $(d \times d)$  matrices unless otherwise stated.

In this paper, we are mainly concerned with statistical procedures for testing whether the observed process  $(\underline{Y}_t, t \in \mathbb{Z})$  is covariance stationary. Hence, the question arises whether the spectral density matrix  $\mathbf{f}(\omega)$  of  $(\underline{X}_t, t \in \mathbb{Z})$  defined in (2.5) has some specific and possibly unique shape if the underlying process  $(\underline{Y}_t, t \in \mathbb{Z})$  is indeed second-order stationary. To answer the

question just posed, it seems convenient not to deal with  $\mathbf{f}(\omega)$  itself, but with some adequately adjusted quantity  $\mathbf{g}(\omega)$  defined below.

Let  $\mathbf{d}(\omega) = \text{diag}(\mathbf{D}_1(\omega), \dots, \mathbf{D}_s(\omega))$  be an  $(sd \times sd)$  diagonal matrix, where  $\mathbf{D}_j(\omega) = e^{-i\frac{j}{s}\omega} \mathbf{I}_d$ ,  $j = 1, \dots, s$  are  $(d \times d)$  diagonal matrices. Now, define  $\mathbf{g}(\omega)$  according to

$$\mathbf{g}(\omega) = \mathbf{d}(\omega)\mathbf{f}(\omega)\mathbf{d}^H(\omega), \quad \omega \in [-\pi, \pi]. \quad (2.6)$$

The  $(sd \times sd)$ -valued function  $\mathbf{g}(\omega)$  is called *modified spectral density* of  $(\underline{X}_t, t \in \mathbb{Z})$  from now on. This quantity has already been used by Troutman (1979) to describe the limiting spectral density of a nonstationary periodic autoregressive model. Similar to equation (2.5),  $(d \times d)$  matrices  $\mathbf{G}_{mn}(\omega)$  are introduced by

$$\mathbf{g}(\omega) = \begin{pmatrix} \mathbf{G}_{11}(\omega) & \mathbf{G}_{12}(\omega) & \cdots & \mathbf{G}_{1s}(\omega) \\ \mathbf{G}_{21}(\omega) & \mathbf{G}_{22}(\omega) & & \mathbf{G}_{2s}(\omega) \\ \vdots & & \ddots & \vdots \\ \mathbf{G}_{s1}(\omega) & \mathbf{G}_{s2}(\omega) & \cdots & \mathbf{G}_{ss}(\omega) \end{pmatrix} \quad (2.7)$$

and due to Hermiticity of  $\mathbf{f}(\omega)$ , this property holds for  $\mathbf{g}(\omega)$  as well, that is,  $\mathbf{G}_{mn}(\omega) = \mathbf{G}_{nm}^H(\omega)$  for all  $m, n$ . In particular, we have

$$\mathbf{G}_{mn}(\omega) = \mathbf{D}_m(\omega)\mathbf{F}_{mn}(\omega)\mathbf{D}_n^H(\omega) = \mathbf{F}_{mn}(\omega)e^{-i\frac{m-n}{s}\omega} \quad (2.8)$$

The following Theorem 2.1 provides exactly the relationship between covariance stationarity of  $(\underline{Y}_t, t \in \mathbb{Z})$ , the autocovariance function  $\mathbf{\Gamma}(h)$  of  $(\underline{X}_t, t \in \mathbb{Z})$  and the modified spectral density  $\mathbf{g}(\omega)$  of  $(\underline{X}_t, t \in \mathbb{Z})$ .

**Theorem 2.1** (Properties of  $(\underline{X}_t, t \in \mathbb{Z})$  under stationarity of  $(\underline{Y}_t, t \in \mathbb{Z})$ ).

Let  $s \geq 2$  and assume  $\sum_{h \in \mathbb{Z}} |\Gamma_{mn}(h)| < \infty$  for all  $m, n = 1, \dots, sd$ . The following assertions (i), (ii) and (iii) are equivalent:

- (i) The  $d$ -variate process  $(\underline{Y}_t, t \in \mathbb{Z})$  in (2.1) is covariance stationary.
- (ii) The autocovariance function  $\mathbf{\Gamma}(h)$  of the  $sd$ -variate process  $(\underline{X}_t, t \in \mathbb{Z})$  in (2.4) fulfills
  - (iia)  $\mathbf{\Gamma}_{m,m+r}(h) = \mathbf{\Gamma}_{n,n+r}(h)$  and  $\mathbf{\Gamma}_{m+r,m}(h) = \mathbf{\Gamma}_{n+r,n}(h)$  for all  $r = 0, \dots, s-1$ ,  $m, n = 1, \dots, s-r$  and  $h \in \mathbb{Z}$ ,
  - (iib)  $\mathbf{\Gamma}_{m+r,m}(h) = \mathbf{\Gamma}_{n,n+s-r}(h+1)$  for all  $r = 1, \dots, s-1$ ,  $m = 1, \dots, s-r$ ,  $n = 1, \dots, r$  and  $h \in \mathbb{Z}$ .
- (iii) The modified spectral density  $\mathbf{g}(\omega)$  of the  $sd$ -variate process  $(\underline{X}_t, t \in \mathbb{Z})$  in (2.7) fulfills
  - (iiia)  $\mathbf{G}_{m,m+r}(\omega) = \mathbf{G}_{n,n+r}(\omega)$  and  $\mathbf{G}_{m+r,m}(\omega) = \mathbf{G}_{n+r,n}(\omega)$  for all  $r = 0, \dots, s-1$ ,  $m, n = 1, \dots, s-r$  and  $\omega \in [-\pi, \pi]$ ,
  - (iiib)  $\mathbf{G}_{m,m+r}(\omega) = \mathbf{G}_{n,n+s-r}^H(\omega)$  and  $\mathbf{G}_{m+r,m}(\omega) = \mathbf{G}_{n,n+s-r}(\omega)$  for all  $r = 0, \dots, s-1$ ,  $m = 1, \dots, s-r$ ,  $n = 1, \dots, r$  and  $\omega \in [-\pi, \pi]$ .

In principle, it is possible to use either the specific structure of the autocovariance function  $\mathbf{\Gamma}(h)$  or of the modified spectral density  $\mathbf{g}(\omega)$  to derive statistical procedures for testing whether the underlying process  $(\underline{Y}_t, t \in \mathbb{Z})$  is covariance stationary. However, the focus in Section 3 of this paper is on a test statistic based on the modified spectral density. For instance, Paparoditis (2000) discussed the advantages of frequency-domain tests based on spectral densities in comparison to time-domain tests based on autocovariances.

Subsequently, Remark 2.1 illustrates the specific shape of  $\mathbf{g}(\omega)$  under stationarity of  $(\underline{Y}_t, t \in \mathbb{Z})$  as stated in Theorem 2.1.

**Remark 2.1** (On  $\mathbf{g}(\omega)$  under covariance stationarity of  $(\underline{Y}_t, t \in \mathbb{Z})$ ).

- (i) Assertion (iiia) of Theorem 2.1 indicates that all block matrices on the principal diagonal of  $\mathbf{g}(\omega)$  are equal. Moreover, all block matrices on one and the same (lower or upper) secondary diagonal of  $\mathbf{g}(\omega)$  are equal.
- (ii) Theorem 2.1 (iiib) establishes a relationship between the block matrices of distinct secondary diagonals of  $\mathbf{g}(\omega)$ . Precisely, the block matrices on the  $r$ th upper (lower) diagonal are equal to the conjugate transpose matrices on the  $(s - r)$ th upper (lower) diagonal, when the principal diagonal is understood as 0th diagonal.
- (iii) Note that, for  $s$  even, result (iiib) of Theorem 2.1 yields, that all  $(d \times d)$  block matrices in the  $\frac{s}{2}$ th secondary diagonal of  $\mathbf{g}(\omega)$  are hermitian themselves. In the case of an univariate underlying time series  $(Y_t, t \in \mathbb{Z})$ , that is  $d = 1$ , this means all entries on the  $\frac{s}{2}$ th secondary diagonal are real-valued.

To simplify notational issues concerning the indices in part (iii) of Theorem 2.1, introduce a more convenient modulo notation that is used throughout the paper from now on. Define

$$\langle n \rangle = (n - 1) \bmod s + 1 \in \{1, \dots, s\}, \quad n \in \mathbb{Z}.$$

This convention is employed in Corollary 2.1 to unify notation and to derive a set of equations that is equivalent to assertion (iii) of Theorem 2.1.

**Corollary 2.1** (Properties of  $\mathbf{g}(\omega)$  under stationarity of  $(\underline{Y}_t, t \in \mathbb{Z})$ ).

Let  $s \geq 2$  and assume  $\sum_{h \in \mathbb{Z}} |\Gamma_{mn}(h)| < \infty$  for all  $m, n = 1, \dots, sd$ . The  $d$ -variate process  $(\underline{Y}_t, t \in \mathbb{Z})$  in (2.1) is covariance stationary if and only if

$$\mathbf{G}_{mn}(\omega) = \mathbf{G}_{\langle m+j \rangle, \langle n+j \rangle}(\omega) \quad (2.9)$$

for all  $m, n = 1, \dots, s$ ,  $j = 0, \dots, s - 1$  and all  $\omega \in [-\pi, \pi]$ . In the following, we write briefly  $\mathbf{G}_{m+j, n+j}(\omega)$  meaning  $\mathbf{G}_{\langle m+j \rangle, \langle n+j \rangle}(\omega)$ .

The following Example 2.1 illustrates the specific shape of the modified spectral density matrix  $\mathbf{g}(\omega)$  of process  $(\underline{X}_t, t \in \mathbb{Z})$  in the situation of an underlying stationary  $d$ -variate  $MA(1)$  model  $(\underline{Y}_t, t \in \mathbb{Z})$  for different number of periods  $s$ .

**Example 2.1** ( $\mathbf{g}(\omega)$  for  $MA(1)$  process  $(\underline{Y}_t, t \in \mathbb{Z})$ ).

Let  $(\underline{Y}_t, t \in \mathbb{Z})$  be a stationary  $d$ -variate  $MA(1)$  process, that is,

$$\underline{Y}_t = \underline{\epsilon}_t + \mathbf{b}\underline{\epsilon}_{t-1}, \quad t \in \mathbb{Z},$$

where  $\underline{\epsilon}_t \sim (0, \Sigma)$  is a  $d$ -variate i.i.d. white noise. The corresponding process  $(\underline{X}_t, t \in \mathbb{Z})$  as defined in (2.3) becomes an  $sd$ -variate  $MA(1)$  process and computing its modified spectral density  $\mathbf{g}_s(\omega)$  for  $s = 2, 3, 4$  results in

$$\begin{aligned} \mathbf{g}_2(\omega) &= \frac{1}{2\pi} \begin{pmatrix} \Sigma + \mathbf{b}\Sigma\mathbf{b}^T & \Sigma\mathbf{b}^T e^{i\frac{1}{2}\omega} + \mathbf{b}\Sigma e^{-i\frac{1}{2}\omega} \\ \mathbf{b}\Sigma e^{-i\frac{1}{2}\omega} + \Sigma\mathbf{b}^T e^{i\frac{1}{2}\omega} & \Sigma + \mathbf{b}\Sigma\mathbf{b}^T \end{pmatrix}, \\ \mathbf{g}_3(\omega) &= \frac{1}{2\pi} \begin{pmatrix} \Sigma + \mathbf{b}\Sigma\mathbf{b}^T & \Sigma\mathbf{b}^T e^{i\frac{1}{3}\omega} & \mathbf{b}\Sigma e^{-i\frac{1}{3}\omega} \\ \mathbf{b}\Sigma e^{-i\frac{1}{3}\omega} & \Sigma + \mathbf{b}\Sigma\mathbf{b}^T & \Sigma\mathbf{b}^T e^{i\frac{2}{3}\omega} \\ \Sigma\mathbf{b}^T e^{i\frac{1}{3}\omega} & \mathbf{b}\Sigma e^{-i\frac{1}{3}\omega} & \Sigma + \mathbf{b}\Sigma\mathbf{b}^T \end{pmatrix}, \\ \mathbf{g}_4(\omega) &= \frac{1}{2\pi} \begin{pmatrix} \Sigma + \mathbf{b}\Sigma\mathbf{b}^T & \Sigma\mathbf{b}^T e^{i\frac{1}{4}\omega} & \mathbf{0}_d & \mathbf{b}\Sigma e^{-i\frac{1}{4}\omega} \\ \mathbf{b}\Sigma e^{-i\frac{1}{4}\omega} & \Sigma + \mathbf{b}\Sigma\mathbf{b}^T & \Sigma\mathbf{b}^T e^{i\frac{1}{4}\omega} & \mathbf{0}_d \\ \mathbf{0}_d & \mathbf{b}\Sigma e^{-i\frac{1}{4}\omega} & \Sigma + \mathbf{b}\Sigma\mathbf{b}^T & \Sigma\mathbf{b}^T e^{i\frac{3}{4}\omega} \\ \Sigma\mathbf{b}^T e^{i\frac{1}{4}\omega} & \mathbf{0}_d & \mathbf{b}\Sigma e^{-i\frac{1}{4}\omega} & \Sigma + \mathbf{b}\Sigma\mathbf{b}^T \end{pmatrix}, \end{aligned}$$

where the  $(d \times d)$  zero matrix is denoted by  $\mathbf{0}_d$ . It can be easily verified that  $\mathbf{g}_s(\omega)$  satisfies in all three cases the properties stated in Theorem 2.1 (iii) and Corollary 2.1, respectively.



Obviously, a periodically stationary process  $(\underline{Y}_t, t \in \mathbb{Z})$  with  $s$  periods and predetermined  $s \geq 2$  defined in (2.1) is covariance stationary if

$$\mathbf{b}_k^{(j)} = \mathbf{b}_k \quad \text{and} \quad \Sigma_j = \Sigma \quad (2.10)$$

for all  $j = 1, \dots, s$  and all  $k \in \mathbb{Z}$ . However, Example 2.2 points out that periodically stationary processes  $(\underline{Y}_t, t \in \mathbb{Z})$  may be covariance stationary without fulfilling (2.10), but also that there exists a representation of  $(\underline{Y}_t, t \in \mathbb{Z})$  satisfying (2.10).

**Example 2.2.**

Let  $(Y_t, t \in \mathbb{Z})$  be a univariate periodically stationary moving average process of order two with  $s = 2$  periods. This is a special case of (2.1) with  $b_k^{(j)} = 0$  for all  $j = 1, 2$  and  $k \in \mathbb{Z} \setminus \{0, 1, 2\}$ . The corresponding process is called PMA(2) process with two periods, briefly. Now, consider the concrete situation

$$\begin{aligned} b_1^{(1)} &= 1, & b_2^{(1)} &= 2, & \Sigma_1 &= 1, \\ b_1^{(2)} &= -2, & b_2^{(2)} &= \frac{1}{2}, & \Sigma_2 &= 4. \end{aligned}$$

By computing its (periodic) autocovariance function  $\Gamma_Y(h, m)$  defined in (2.2), we get

$$\Gamma_Y(h, 1) = \Gamma_Y(h, 2) = \Gamma_Y(h)$$

for all  $h \in \mathbb{Z}$ , that is,  $(Y_t, t \in \mathbb{Z})$  is covariance stationary. Actually, regarding their autocovariance structure,  $(Y_t, t \in \mathbb{Z})$  and the MA(2) process  $(\tilde{Y}_t, t \in \mathbb{Z})$  with

$$\tilde{Y}_t = \tilde{\epsilon}_t + \tilde{b}_2 \tilde{\epsilon}_{t-2}, \quad t \in \mathbb{Z}$$

and i.i.d. white noise  $\tilde{\epsilon}_t \sim (0, \frac{2}{\tilde{b}_2})$  where  $\tilde{b}_2 \in \{\frac{\sqrt{65+9}}{4}, -\frac{\sqrt{65+9}}{4}\}$  are indistinguishable. Due to relation (2.3), this property holds also for the corresponding bivariate processes  $(\underline{X}_t, t \in \mathbb{Z})$  and  $(\tilde{\underline{X}}_t, t \in \mathbb{Z})$  and this causes their spectral densities  $\mathbf{f}(\omega)$  and  $\tilde{\mathbf{f}}(\omega)$  to be identical.

### 3. THE TEST STATISTIC AND ASYMPTOTIC RESULTS

#### 3.1. Construction of the test statistic.

Let  $s \geq 2$ . Suppose we have  $d$ -dimensional data  $\underline{Y}_1, \dots, \underline{Y}_N$  with  $N = sM$ ,  $M \in \mathbb{N}$  generated by (2.1) at hand and we are interested in testing for stationarity of the process  $(\underline{Y}_t, t \in \mathbb{Z})$ , that is, the null hypothesis of interest is

$$H_0: (\underline{Y}_t, t \in \mathbb{Z}) \text{ is covariance stationary}$$

against the alternative

$$H_1: (\underline{Y}_t, t \in \mathbb{Z}) \text{ is not covariance stationary, but periodically stationary with } s \text{ periods.}$$

To motivate a test statistic for  $H_0$  against  $H_1$  suppose that the spectral density matrix  $\mathbf{f}(\omega)$  of the corresponding  $sd$ -variate process  $(\underline{X}_t, t \in \mathbb{Z})$  is known for all frequencies  $\omega$ . Hence, we are able to compute  $\mathbf{g}(\omega)$  as well, because  $\mathbf{d}(\omega)$  is deterministic and known. Under  $H_0$ , equation (2.9) in Corollary 2.1 holds true and is equivalent to

$$\|\mathbf{G}_{mn}(\omega) - \frac{1}{s} \sum_{j=0}^{s-1} \mathbf{G}_{m+j, n+j}(\omega)\|^2 = 0 \quad (3.1)$$

for all  $\omega \in [-\pi, \pi]$  and all  $m, n = 1, \dots, s$ , where for a matrix  $A$ ,  $\|A\|$  denotes its Euclidean matrix norm. The previous equation suggests that a way to test the null hypothesis is to test for



the squared and normed expression in (3.1) to be equal to zero on the whole interval  $\omega \in [-\pi, \pi]$  and for all  $m, n$ . Equivalently, integrating and summing-up (3.1) gives

$$\int_{-\pi}^{\pi} \sum_{m,n=1}^s \|\mathbf{G}_{mn}(\omega) - \frac{1}{s} \sum_{j=0}^{s-1} \mathbf{G}_{m+j,n+j}(\omega)\|^2 d\omega = 0. \quad (3.2)$$

Because  $\mathbf{g}(\omega)$  is unknown in general, let  $\widehat{\mathbf{G}}_{mn}(\omega)$  be the canonical nonparametric estimate for  $\mathbf{G}_{mn}(\omega)$  obtained by smoothing the so-called *modified periodogram matrix*

$$\widehat{\mathbf{I}}_M(\omega) = \mathbf{d}(\omega) \mathbf{I}_M(\omega) \mathbf{d}^H(\omega), \quad (3.3)$$

where  $\mathbf{I}_M(\omega) = \underline{J}_M(\omega) \underline{J}_M^H(\omega)$  is the usual periodogram matrix based on  $\underline{X}_1, \dots, \underline{X}_M$ ,

$$\underline{J}_M(\omega) = \frac{1}{\sqrt{2\pi M}} \sum_{t=1}^M \underline{X}_t e^{-it\omega}$$

denotes the multiple discrete Fourier transform and  $\mathbf{d}(\omega)$  is defined before equation (2.6) in Section 2. Precisely, define

$$\widehat{\mathbf{g}}(\omega) = \frac{1}{M} \sum_{k=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} K_h(\omega - \omega_k) \widehat{\mathbf{I}}_M(\omega_k) = \begin{pmatrix} \widehat{\mathbf{G}}_{mn}(\omega) \\ m, n = 1, \dots, s \end{pmatrix} = \begin{pmatrix} \widehat{g}_{mn}(\omega) \\ m, n = 1, \dots, sd \end{pmatrix}, \quad (3.4)$$

where  $\lfloor x \rfloor$  is the integer part of  $x \in \mathbb{R}$  and  $K$  is a nonnegative symmetric kernel function with compact support  $[-\pi, \pi]$  satisfying  $\frac{1}{2\pi} \int_{-\pi}^{\pi} K(x) dx = 1$ ,  $h$  is the bandwidth and  $K_h(\cdot) = \frac{1}{h} K(\frac{\cdot}{h})$ . Recall that  $N$   $d$ -dimensional observations correspond to an  $sd$ -dimensional realization of length  $M$ . Because inference is based on properties of the  $sd$ -variate stationary process, the test statistic below is considered to depend on  $M$  instead of  $N$ .

Now, replacing all unknown quantities in (3.2) by their kernel estimates as derived in (3.4), defining  $\widetilde{\mathbf{G}}_{mn}(\omega) = \frac{1}{s} \sum_{j=0}^{s-1} \widehat{\mathbf{G}}_{m+j,n+j}(\omega)$  and introducing the notation

$$\widetilde{\mathbf{g}}(\omega) = \begin{pmatrix} \widetilde{\mathbf{G}}_{mn}(\omega) \\ m, n = 1, \dots, s \end{pmatrix} = \begin{pmatrix} \widetilde{g}_{mn}(\omega) \\ m, n = 1, \dots, sd \end{pmatrix}, \quad (3.5)$$

results in an  $L_2$ -type test statistic

$$S_M = Mh^{\frac{1}{2}} \int_{-\pi}^{\pi} \|\widehat{\mathbf{g}}(\omega) - \widetilde{\mathbf{g}}(\omega)\|^2 d\omega, \quad (3.6)$$

where the use of the inflation factor  $Mh^{\frac{1}{2}}$  is due to the variance of  $S_M$  as shown in the proof of Theorem 3.1. Observe that the  $(d \times d)$  block entries  $\widetilde{\mathbf{G}}_{mn}(\omega)$  of  $\widetilde{\mathbf{g}}(\omega)$  are average values computed over all blocks of  $\widehat{\mathbf{g}}(\omega)$  that estimate equal quantities under the null hypothesis.

For computational reasons, it is sometimes preferred to avoid matrix norms in the representation of  $S_M$  in (3.6). Hence, note that  $\widetilde{g}_{mn}(\omega) = \frac{1}{s} \sum_{j=0}^{s-1} \widehat{g}_{m+dj,n+dj}(\omega)$  and

$$S_M = Mh^{\frac{1}{2}} \int_{-\pi}^{\pi} \sum_{m,n=1}^{sd} |\widehat{g}_{mn}(\omega) - \widetilde{g}_{mn}(\omega)|^2 d\omega. \quad (3.7)$$

At this point, it is worth noting that ratio-type statistics as used in Paparoditis (2000, 2005) for goodness-of-fit testing are not suitable in this situation. This is because we deal exclusively with nonparametric kernel spectral density estimation compared to a combined parametric and nonparametric estimation scheme in the goodness-of-fit setup.

### 3.2. Assumptions and asymptotic results.

In order to apply the test statistic  $S_M$ , we need its distribution under the null hypothesis. To derive the asymptotic limit of this distribution the following assumptions are imposed.

- (A) The process  $(\underline{\epsilon}_t, t \in \mathbb{Z})$  has finite absolute moments of order  $16 + \delta$  for some  $\delta > 0$ ,  $\sum_{k=-\infty}^{\infty} |k| |B_{k;m,n}| < \infty$  for all  $m, n = 1, \dots, s$  and the  $sd$ -variate process  $(\underline{X}_t, t \in \mathbb{Z})$  is supposed to be absolutely regular [cf. Pham and Tran (1985)].
- (K) The function  $K$  denotes a nonnegative, bounded and Lipschitz continuous kernel with compact support  $[-\pi, \pi]$  satisfying  $\frac{1}{2\pi} \int_{-\pi}^{\pi} K(u) du = 1$ .
- (B) The bandwidth  $h$  satisfies  $h \rightarrow 0$  and  $Mh \rightarrow \infty$  as  $M \rightarrow \infty$ .

The moment assumptions and absolute regularity of  $(\underline{X}_t, t \in \mathbb{Z})$  in (A) are required to apply a central limit theorem of Gao and Hong (2008) to  $S_M$  and the summability assumption guarantees  $E(I_M(\omega)) = \mathbf{f}(\omega) + O(\frac{1}{M})$  uniformly in  $\omega$ , for instance.

A unifying contribution on the topic of testing nonparametric and semiparametric hypothesis in the frequency domain is the paper of Eichler (2008), where a general asymptotic theory under certain mixing conditions has been developed. Since this paper deals with linear processes, asymptotic normality of the proposed test statistic  $S_M$  is proved under a set of conditions different to the assumptions used in Eichler (2008).

The following theorem deals with the asymptotic normal distribution of the test statistic  $S_M$  under the null hypothesis of process  $(\underline{Y}_t, t \in \mathbb{Z})$  defined in (2.1) to be covariance stationary. First, we set

$$A_K = \frac{1}{2\pi} \int_{-\pi}^{\pi} K^2(v) dv, \quad B_K = \frac{1}{\pi^2} \int_{-2\pi}^{2\pi} \left( \int_{-\pi}^{\pi} K(x) K(x+z) dx \right)^2 dz$$

and

$$\kappa(t) = \begin{cases} s-1, & t=0 \\ -1, & t=\frac{s}{2} \\ -2, & \text{otherwise} \end{cases}.$$

Also, we introduce the notation  $P(\mathbf{F}, t, \omega) = \text{tr}(\mathbf{F}_{1,t}(\omega))$  and  $Q(\mathbf{F}, n, t, \omega) = \text{tr}(\mathbf{F}_{1,n}(\omega) \mathbf{F}_{1,n+t}^H(\omega))$ , where  $\text{tr}(\mathbf{A}) = \sum_{k=1}^d a_{kk}$  is the trace of a  $(d \times d)$  matrix  $\mathbf{A}$ .

**Theorem 3.1** (Asymptotic null distribution of  $S_M$ ).

Suppose that assumptions (A), (K) and (B) are fulfilled. If  $H_0$  is true, then

$$S_M - \mu_h \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2)$$

as  $M \rightarrow \infty$ , where  $\xrightarrow{\mathcal{D}}$  denotes convergence in distribution,

$$\mu_h = h^{-\frac{1}{2}} A_K \int_{-\pi}^{\pi} s \sum_{t=1}^{\lfloor \frac{s}{2} \rfloor} \kappa(t) |P(\mathbf{F}, t, \omega)|^2 d\omega \quad (3.8)$$

and

$$\tau^2 = B_K \int_{-\pi}^{\pi} s \sum_{t=1}^{\lfloor \frac{s}{2} \rfloor} \kappa(t) \left| \sum_{n=1}^s Q(\mathbf{F}, n, t, \omega) \right|^2 d\omega. \quad (3.9)$$

Based on Theorem 3.1 and for  $\alpha \in (0, 1)$  a test for the null hypothesis  $H_0$  against the alternative  $H_1$  of asymptotic level  $\alpha$  is obtained by rejecting  $H_0$  if

$$\frac{S_M - \mu_h}{\tau} \geq u_{1-\alpha}, \quad (3.10)$$

where  $\tau = \sqrt{\tau^2}$  and  $u_{1-\alpha}$  is the  $(1 - \alpha)$ -quantile of the standard normal distribution.

Note that  $\mu_h = O(h^{-\frac{1}{2}})$ , that is, the centralizing term  $\mu_h$  tends to infinity with increasing sample size. It is well-known that this property is fulfilled for  $L_2$ -type statistics in general [cf. Härdle and Mammen (1993), Paparoditis (2000)]. Moreover, it is worth noting that the test statistic  $S_M$  has to be evaluated using the (estimated) modified spectral density  $\mathbf{g}(\omega)$ , but its asymptotic normal distribution may be expressed using exclusively the usual spectral density  $\mathbf{f}(\omega)$ .

The following example illustrates how the complicated structure of  $\mu_h$  and  $\tau^2$  derived in Theorem 3.1 simplifies in a special case.

**Example 3.1** ( $\mu_h$  and  $\tau^2$  for  $d = 1$  and  $s = 2$ ).

Let  $d = 1$  and  $s = 2$ . Under the assumptions of Theorem 3.1, it holds

$$\mu_h = h^{-\frac{1}{2}} A_K \int_{-\pi}^{\pi} 2|f_{11}(\omega)|^2 (1 - C_{12}(\omega)) d\omega$$

and

$$\tau^2 = B_K \int_{-\pi}^{\pi} 2|f_{11}(\omega)|^4 (1 - C_{12}(\omega))^2 d\omega,$$

where  $C_{jk}(\omega) = \frac{|f_{jk}(\omega)|^2}{f_{jj}(\omega)f_{kk}(\omega)}$  is the squared coherence between the two components  $j$  and  $k$  of  $(\underline{X}_t, t \in \mathbb{Z})$  [cf. Hannan (1970)].

A computationally more attractive version of  $S_M$  is given by its discretization

$$\widehat{S}_M = 2\pi h^{\frac{1}{2}} \sum_{j=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} \|\widehat{\mathbf{g}}(\omega_j) - \widetilde{\mathbf{g}}(\omega_j)\|^2,$$

which is obtained by approximating the integral in (3.6) by its corresponding Riemann sum. This discretized version is asymptotically equivalent to  $S_M$ , but its asymptotic distribution derived under  $H_0$  in Theorem 3.1 still depends through  $\mu_h$  and  $\tau^2$  on the unknown quantities  $\mathbf{F}_{mn}(\omega)$ . By default, these may be estimated nonparametrically and replacing them by their canonical estimates  $\widehat{\mathbf{F}}_{mn}(\omega)$  and approximating all unknown integrals in (3.8) and (3.9) by their Riemann sums does not affect the asymptotic distribution of  $S_M$  either. This can be proved by using very similar arguments to those employed for proving Theorem 3.1 and these considerations result in the following Corollary 3.1.

**Corollary 3.1** (Asymptotic null distribution of  $\widehat{S}_M$ ).

Suppose the assumptions of Theorem 3.1 are fulfilled and  $H_0$  is true. Then it holds

$$\frac{\widehat{S}_M - \widehat{\mu}_h}{\widehat{\tau}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$$

as  $M \rightarrow \infty$ , where

$$\widehat{\mu}_h = h^{-\frac{1}{2}} A_K \frac{2\pi}{M} \sum_{j=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} \left( s \sum_{t=1}^{\lfloor \frac{s}{2} \rfloor} \kappa(t) \left| P(\widehat{\mathbf{F}}, t, \omega_j) \right|^2 \right)$$

and

$$\hat{\tau}^2 = B_K \frac{2\pi}{M} \sum_{j=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} \left( s \sum_{t=1}^{\lfloor \frac{s}{2} \rfloor} \kappa(t) \left| \sum_{n=1}^s Q(\hat{\mathbf{F}}, n, t, \omega_j) \right|^2 \right)$$

Therefore, we reject the null hypothesis  $H_0$  if

$$\frac{\hat{S}_M - \hat{\mu}_h}{\hat{\tau}} \geq u_{1-\alpha}, \quad (3.11)$$

which seems to be of more practical relevance compared to (3.10) due to computational convenience. Actually, because of Theorem 2.1 (iiib), the representations of  $\mu_h(K)$  and  $\tau^2(K)$  in (3.8) and (3.9) (and of its approximations  $\hat{\mu}_h(K)$  and  $\hat{\tau}^2(K)$ , as well) contain some redundancy, but for notational reasons it seems more convenient to allow for some redundancy in this context.

Before Theorem 3.2 below characterizes the behaviour of  $S_M$  under the alternative, the following remark discusses the issue of estimating  $\mu_h$  and  $\tau^2$ .

**Remark 3.1** (On computing  $\hat{\mu}_h$  and  $\hat{\tau}^2$ ).

Observe, to estimate  $\mu_h$  and  $\tau^2$  nonparametrically, it is not necessary to compute additional quantities. On the one hand, this is because replacing  $\mathbf{F}_{mn}(\omega)$  in (3.8) and (3.9) by  $\mathbf{G}_{mn}(\omega)$  does not alter  $\mu_h$  and  $\tau^2$ . And, under  $H_0$ ,  $\tilde{\mathbf{G}}_{mn}(\omega)$  is some kind of a pooled estimate for  $\mathbf{G}_{mn}(\omega)$ , which has to be computed for the test statistic  $S_M$  anyway and uses most information provided by the data, on the other hand.

The following Theorem 3.2 proves  $S_M$  (and  $\hat{S}_M$ ) to be an omnibus test that has positive power against any alternative belonging to  $H_1$ .

**Theorem 3.2** (Omnibus property of  $S_M$ ).

Let the assumptions of Theorem 3.1 be true and assume that  $H_0$  is wrong, that is,  $(\underline{Y}_t, t \in \mathbb{Z})$  is not covariance stationary, but periodically stationary with  $s$  periods,  $s \geq 2$ . If  $M \rightarrow \infty$ , then

$$M^{-1} h^{-\frac{1}{2}} S_M \rightarrow \int_{-\pi}^{\pi} \|\mathbf{g}(\omega) - \mathbf{g}_1(\omega)\|^2 d\omega$$

in probability, where  $\mathbf{g}_1(\omega)$  denotes the limit in probability of  $\tilde{\mathbf{g}}(\omega)$  under the alternative  $H_1$ .

Since under the alternative  $\mathbf{g}(\cdot) - \mathbf{g}_1(\cdot) \neq \mathbf{0}$  due to Theorem 2.1, continuity of  $\mathbf{g}(\cdot) - \mathbf{g}_1(\cdot)$  implies

$$\int_{-\pi}^{\pi} \|\mathbf{g}(\omega) - \mathbf{g}_1(\omega)\|^2 d\omega > 0.$$

Therefore,  $S_M$  is an omnibus test that has positive power against any alternative belonging to the alternative  $H_1$ .

### 3.3. Testing the null hypothesis of periodic stationarity.

In the setup of an underlying periodically stationary model as, for instance, defined in equation (2.1), typically, the test situation  $H_0$  against  $H_1$  is considered exclusively [cf. Vecchia and Balcerini (1991), Ursu and Duchesne (2009)].

However, it is also of considerable relevance to ask whether the process defined in (2.1) is periodically stationary with a certain number of periods  $s_0 < s$ . For instance, Franses and Paap (2004) have studied the case of quaterly data, that is,  $s = 4$  in detail and, in this setup, the canonical question arises whether the true periodicity  $s_0$  is possibly 2 instead of 4. A test procedure restricted to simple hypothesis and finite number of autocovariance lags that is applicable

to this situation is proposed in Lenart, Leśkow and Synowiecki (2008).

Now, suppose  $s \geq 2$  in (2.1) and assume that the periodicity of interest  $s_0$  satisfies  $s > s_0 \geq 2$  and, additionally,  $s' = \frac{s}{s_0} \in \mathbb{N}$ . Consider the test situation

$H_0^{(s_0)}$ :  $(\underline{Y}_t, t \in \mathbb{Z})$  is periodically stationary with  $s_0$  periods

against the alternative

$H_1^{(s_0)}$ :  $(\underline{Y}_t, t \in \mathbb{Z})$  is periodically stationary with  $s$  periods, but *not* with  $s_0$  periods.

To transfer the theory developed in Section 3.2 above to test  $H_0^{(s_0)}$  against  $H_1^{(s_0)}$ , observe that  $(\underline{Y}_t, t \in \mathbb{Z})$  in (2.1) is periodically stationary with  $s_0$  periods if and only if the  $d'$ -dimensional linear process  $(\underline{Z}_t, t \in \mathbb{Z})$  with  $d' = s_0 d$  and

$$\underline{Z}_{j+s'T} = \begin{pmatrix} \underline{Y}_{1+(j-1)s_0+sT} \\ \underline{Y}_{2+(j-1)s_0+sT} \\ \vdots \\ \underline{Y}_{s_0+(j-1)s_0+sT} \end{pmatrix}, \quad j = 1, \dots, s', \quad T \in \mathbb{Z}, \quad (3.12)$$

is covariance stationary. Therefore, the test situation above may be reformulated and is equivalent to

$H_0^{(s_0)}$ :  $(\underline{Z}_t, t \in \mathbb{Z})$  is covariance stationary

against the alternative

$H_1^{(s_0)}$ :  $(\underline{Z}_t, t \in \mathbb{Z})$  is *not* covariance stationary, but periodically stationary with  $s'$  periods.

Hence, testing  $H_0^{(s_0)}$  against  $H_1^{(s_0)}$  fits into the setup of Section 3.1 and the test statistic  $S_M$  derived originally in Section 3.2 for testing  $H_0$  against  $H_1$  may be used in this situation as well. Observe, this represents an extension of the previous literature that is typically concerned with testing for stationarity in periodic stationary models or that is restricted to simple hypothesis and finite number of lags of the autocovariance function.

#### 4. SIMULATION STUDIES

In this section, the performance of the test statistic proposed in Section 3 is illustrated by means of simulations. Suppose we want to test the null hypothesis  $H_0$  of an underlying stationary process  $(\underline{Y}_t, t \in \mathbb{Z})$  against the alternative  $H_1$  of the process being periodically stationary with period  $s = 2$ . To investigate the behaviour of the test statistic under the null, we consider univariate realizations ( $d = 1$ ) of length  $N = 100$  from an  $MA(1)$  model

$$Y_t = 0.5\epsilon_{t-1} + \epsilon_t, \quad t \in \mathbb{Z}$$

with i.i.d.  $\epsilon_t \sim \mathcal{N}(0, 1)$ , which we call Model I in the following.

To analyze the power of the test procedure under the alternative, we consider observations from two periodic  $MA(1)$  models

$$\begin{aligned} Y_{1+2T} &= 0.3\epsilon_{1+2T-1} + \epsilon_{1+2T}, \\ Y_{2+2T} &= 0.7\epsilon_{2+2T-1} + \epsilon_{2+2T} \end{aligned}$$

with i.i.d.  $\epsilon_t \sim \mathcal{N}(0, 1)$  which we call Model IIa and

$$\begin{aligned} Y_{1+2T} &= 0.5\epsilon_{1+2T-1} + \epsilon_{1+2T}, \\ Y_{2+2T} &= 0.5\epsilon_{2+2T-1} + \epsilon_{2+2T} \end{aligned}$$

with independently and normally distributed white noise  $(\epsilon_t, t \in \mathbb{Z})$  such that  $Var(\epsilon_{j+2T}) = \sigma_j^2$ ,  $j = 1, 2$  with  $\sigma_1 = 0.8$  and  $\sigma_2 = 1.2$  which is indicated as Model IIb. Realizations of Model I, IIa

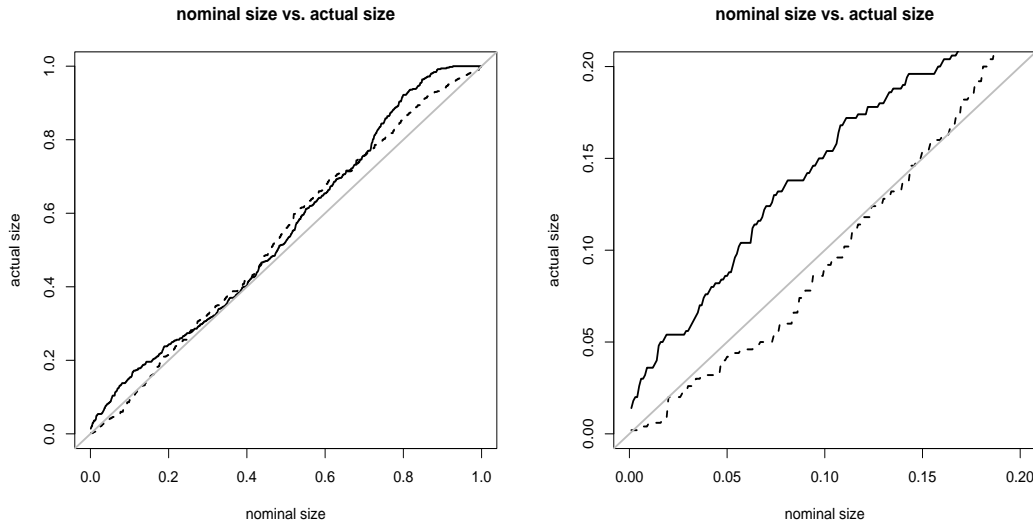


FIGURE 2. P-value plots for Model I on  $[0, 1] \times [0, 1]$  (left panel) and on  $[0, 0.2] \times [0, 0.2]$  (right panel) using critical values from the CLT (solid line) and from the hybrid bootstrap (dashed line) with target indicated by a grey solid line.

and IIb are shown in Figure 1. With  $d = 1$ , observe that Model I is a special case of Example 2.1 and Example 3.1, where the modified spectral density  $\mathbf{g}_2(\omega)$  of the corresponding bivariate stationary process and the asymptotic distribution of the test statistic  $S_M$  under  $H_0$  are given, respectively.

Note that  $N = 100$  univariate observations yield to bivariate time series data of length  $M = 50$  which is used to estimate the modified spectral density matrix  $\mathbf{g}(\omega)$  via  $\hat{\mathbf{g}}(\omega)$  and to compute the related quantity  $\tilde{\mathbf{g}}(\omega)$  for evaluation of the test statistic  $S_M$ . In doing so, we have chosen the bandwidth  $h = 0.3$  and the Bartlett-Priestley kernel has been used in all simulations.

Among others, Paparoditis (2000, 2005) pointed out that weak convergence of  $L_2$ -type statistics of this kind to asymptotic normal distributions is very slow in general. In particular, this holds true for the CLTs presented in Theorem 3.1 and in Corollary 3.1, respectively. Therefore, an appropriate bootstrap technique should be used to construct a bootstrap version of the test that hopefully shows more reasonable behaviour in small sample situations. To get a bootstrap test that has much power, we have to mimic its distribution under  $H_0$  even if the alternative  $H_1$  is true. In particular, we have to use a bootstrap that works asymptotically for kernel spectral density estimators when the underlying process belongs to the general class of linear time series.

Recently, Jentsch and Kreiss (2010) introduced the hybrid bootstrap as an extension of the

	Model I		
nominal size	0.010	0.050	0.100
actual size (CLT)	0.036	0.086	0.152
actual size (hybrid)	0.006	0.042	0.092

TABLE 1. Comparison of nominal size and actual size for Model I.

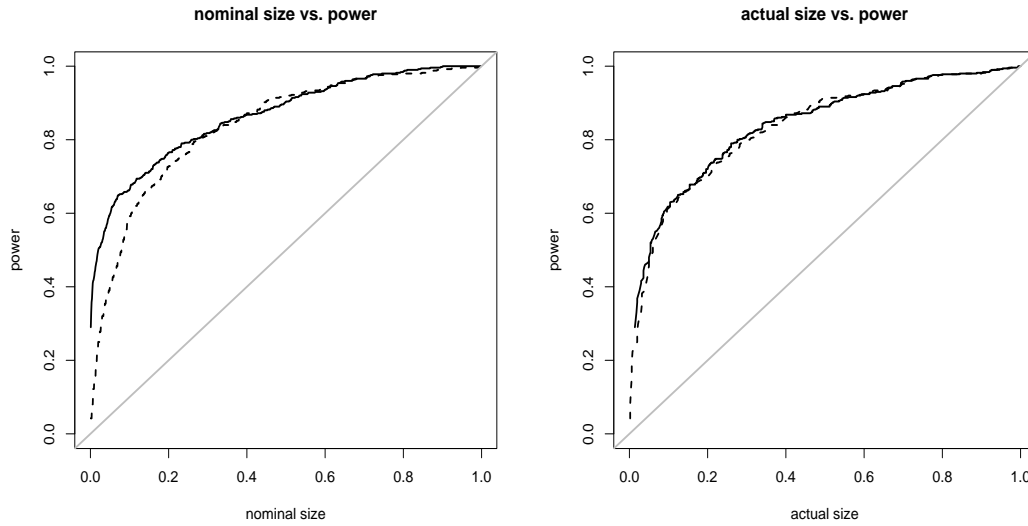


FIGURE 3. Size power curves for Model IIa with nominal size vs. power (left panel) and actual size from Figure 2 vs. power (right panel) using critical values from the CLT (solid line) and from the hybrid bootstrap (dashed line).

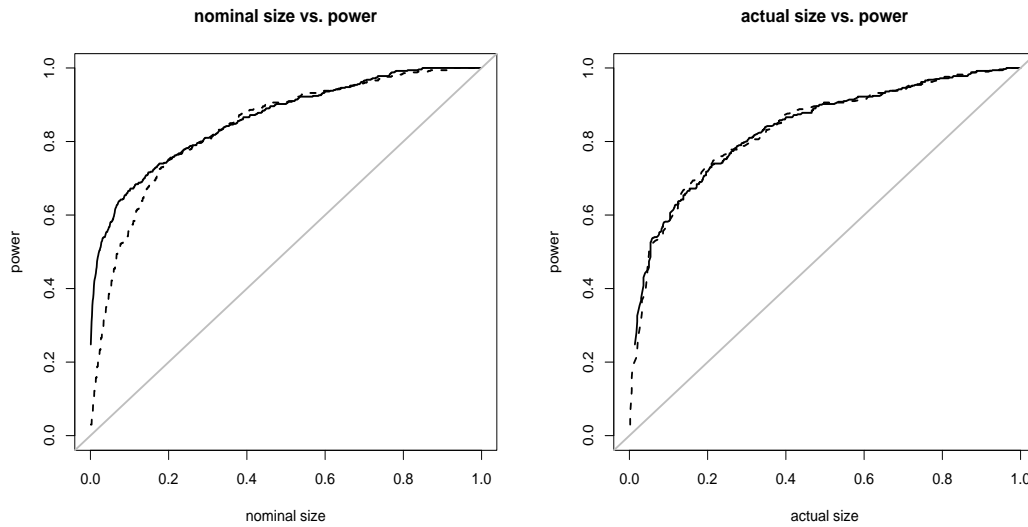


FIGURE 4. Size power curves for Model IIb with nominal size vs. power (left panel) and actual size from Figure 2 vs. power (right panel) using critical values from the CLT (solid line) and from the hybrid bootstrap (dashed line).

autoregressive-aided periodogram bootstrap proposed by Kreiss and Paparoditis (2003). They proved not only validity of their proposal for kernel spectral density estimates for linear processes in general, but they also extended this result to the multivariate case. For this reason, it is possible to apply this bootstrap technique if the underlying processes  $(\underline{Y}_t, t \in \mathbb{Z})$  is multivariate, that is  $d \geq 2$ , as well. Moreover, the hybrid bootstrap may be applied in the situation of



	Model IIa			Model IIb		
nominal size	0.010	0.050	0.100	0.010	0.050	0.100
power (CLT)	0.430	0.598	0.664	0.420	0.574	0.666
power (hybrid)	0.154	0.406	0.592	0.122	0.402	0.564

TABLE 2. Comparison of nominal size and power for Model IIa and IIb.

	Model IIa			Model IIb		
actual size	0.010	0.050	0.100	0.010	0.050	0.100
power (CLT)	NA	0.480	0.618	NA	0.474	0.584
power (hybrid)	0.224	0.478	0.616	0.188	0.500	0.584

TABLE 3. Comparison of actual size and power for Model IIa and IIb.

Section 3.3 as well, where  $H_0^{(s_0)}$  against  $H_1^{(s_0)}$  is tested. The (univariate) hybrid bootstrap test procedure of level  $\alpha \in (0, 1)$  may be summarized as follows:

- Step 1: Fit an  $AR(p)$ -model of some (arbitrarily chosen) order  $p \in \mathbb{N}$  to the data  $Y_1, \dots, Y_N$  and use a residual bootstrap to obtain bootstrap replicates  $Y_1^+, \dots, Y_N^+$ .
- Step 2: Compute the discrete Fourier transform (DFT)  $J_N^+(\omega)$  based on  $Y_1^+, \dots, Y_N^+$  and a non-parametric correction term  $\tilde{q}(\omega)$  at the Fourier frequencies  $\omega_j = 2\pi \frac{j}{N}$ ,  $j = 1, \dots, N$ .
- Step 3: Compute the inverse DFT of the *corrected* DFT  $\tilde{q}(\omega_1)J_N^+(\omega_1), \dots, \tilde{q}(\omega_N)J_N^+(\omega_N)$  to obtain bootstrap observations  $Y_1^*, \dots, Y_N^*$  according to

$$Y_t^* = \sqrt{\frac{2\pi}{N}} \sum_{j=1}^N \tilde{q}(\omega_j) J_N^+(\omega_j) e^{it\omega_j}, \quad t = 1, \dots, N.$$

- Step 4: Compute the bootstrap test  $S_M^*$  based on  $Y_1^*, \dots, Y_N^*$ .
- Step 5: Repeat the Steps 1-4 above  $B$  times and take the  $(1 - \alpha)$ -quantile of the empirical distribution of  $S_{M,1}^*, \dots, S_{M,B}^*$  to get the  $\alpha$ -level bootstrap critical value  $c_{M,\alpha}^*$ .
- Step 5: Finally, reject the null hypothesis if  $S_M \geq c_{M,\alpha}^*$ .

The idea behind the hybrid bootstrap is to use a parametric (autoregressive) fit to capture the main dependence features of the data and to apply a nonparametric correction afterwards in the frequency domain to mimic as much as possible the dependence structure in the data. Compare Jentsch and Kreiss (2010) for a detailed discussion and the choice of  $\tilde{q}(\omega)$ , in particular.

To apply the hybrid bootstrap, some parameters have to be assessed. In our simulation study, we have chosen the autoregressive order  $p = 1$  for the residual bootstrap in Step 1 and the smoothing parameter  $h_b$  implicitly used for computation of  $\tilde{q}(\omega)$  is chosen to be equal to the bandwidth  $h$  for computing  $S_M$ , that is  $h_b = h = 0.3$ , and again the Bartlett-Priestley kernel has been employed. Further, the bootstrap approximation of the distribution of  $S_M$  is based on  $B = 300$  bootstrap replications.

To examine the behaviour of the proposed test statistic,  $T = 500$  data sets have been simulated for all three models under consideration. The results under the null hypothesis of stationarity using Model I are presented in terms of p-value plots in Figure 2. The power behaviour of the test under the alternative is illustrated in terms of size power curves in Figure 3 for Model IIa and in Figure 4 for Model IIb.

The p-value plots in Figure 2 illustrate that the test based on critical values from the CLT in Theorem 3.1 tends to overreject the null hypothesis systematically. A comparison of both curves in Figure 2 (left panel) demonstrates the gain using critical values from the hybrid bootstrap. In particular for small nominal sizes  $\alpha \in [0, 0.2]$  which appear to be crucial for testing purposes, the hybrid bootstrap test does not overreject the null hypothesis anymore as illustrated in the right panel of Figure 2. For this reason, it seems to be unfair to compare just the usual size-power curves as done in the left panels of Figure 3 and of Figure 4, respectively. Therefore, we present also *modified* size-power curves that use actual sizes instead of nominal sizes on the horizontal axis in the right panels of Figure 3 and of Figure 4. Actually, these plots show a very similar shape. The desired effect of the hybrid bootstrap to help the test to hold some commonly used levels is shown in Table 1. Also, the relation of nominal (actual) size and power for the test and the bootstrap test are illustrated in Table 2 (Table 3) for these levels.

In summary, the test based on critical values obtained from the CLT appears to have good power in both Models IIa and IIb, but tends to overreject the null hypothesis in Model I particularly for small nominal size. As pointed out above, the hybrid bootstrap aided test procedure holds the nominal size more accurately under the null and a comparison of modified size-power curves in the right panels of Figure 3 and Figure 4, respectively, demonstrates that the bootstrap versions have about the same power as the corresponding tests based on the CLT.

## 5. PROOFS

### Proof of Theorem 2.1.

Let  $s \geq 2$ . The periodically stationary process  $(\underline{Y}_t, t \in \mathbb{Z})$  defined in (2.1) is covariance stationary if and only if its autocovariance function  $\mathbf{\Gamma}(h, m)$  introduced in (2.2) does not depend on  $m$ , that is,

$$\mathbf{\Gamma}_Y(h, 1) = \mathbf{\Gamma}_Y(h, 2) = \dots = \mathbf{\Gamma}_Y(h, s)$$

for all  $h \in \mathbb{Z}$ . The following equations [cf. for instance Ursu and Duchesne (2009)]

$$\begin{aligned} \mathbf{\Gamma}(h) &= \begin{pmatrix} \mathbf{\Gamma}_{mn}(h) \\ m, n = 1, \dots, s \end{pmatrix} = \begin{pmatrix} \mathbf{\Gamma}_Y(sh + m - n, m) \\ m, n = 1, \dots, s \end{pmatrix}, \\ \mathbf{\Gamma}(h + 1) &= \begin{pmatrix} \mathbf{\Gamma}_{mn}(h + 1) \\ m, n = 1, \dots, s \end{pmatrix} = \begin{pmatrix} \mathbf{\Gamma}_Y(sh + s + m - n, m) \\ m, n = 1, \dots, s \end{pmatrix} \end{aligned} \quad (5.1)$$

for  $h \in \mathbb{Z}$  establish a relationship between the autocovariance functions  $\mathbf{\Gamma}(h)$  and  $\mathbf{\Gamma}_Y(h, m)$  of both processes  $(\underline{Y}_t, t \in \mathbb{Z})$  and  $(\underline{X}_t, t \in \mathbb{Z})$ . It can be easily seen that  $(\underline{Y}_t, t \in \mathbb{Z})$  is covariance stationary if and only if assertion (ii) of the theorem is satisfied.

To prove the second claimed equivalence, the multivariate inversion formula together with equations (2.6) and (5.1) yields

$$\mathbf{g}(\omega) = \begin{pmatrix} \mathbf{G}_{mn}(\omega) \\ m, n = 1, \dots, s \end{pmatrix} = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \begin{pmatrix} \mathbf{\Gamma}_Y(sh + m - n, m) e^{-i \frac{sh+m-n}{s} \omega} \\ m, n = 1, \dots, s \end{pmatrix}. \quad (5.2)$$

Finally, comparison of coefficients in (5.2) gives that  $(\underline{Y}_t, t \in \mathbb{Z})$  is covariance stationary if and only if assertion (iii) is fulfilled, which concludes this proof.  $\square$

**Proof of Theorem 3.1.**

To prove the central limit theorem for  $S_M$  under  $H_0$ , it is convenient to deal with its entry-wise representation in (3.7). First, let  $\widehat{I}_{m,n,j;k} = \widehat{I}_{m+dj,n+dj}(\omega_k)$ ,  $g_{m,n,j_1,j_2;k} = g_{m+dj_1,n+dj_2}(\omega_k)$  and

$$\Delta^{(k)}(j_1, \dots, j_k) = \prod_{w=1}^k (1 - s\delta_{j_w,0}), \quad k = 1, \dots, 4,$$

where  $\delta_{j_0} = 1$  if  $j = 0$  and  $\delta_{j_0} = 0$  otherwise. With this notation, (3.4) and (3.5) yield

$$\widehat{g}_{mn}(\omega) - \widetilde{g}_{mn}(\omega) = \frac{1}{M} \sum_{k=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} K_h(\omega - \omega_k) \left( -\frac{1}{s} \sum_{j=0}^{s-1} \Delta^{(1)}(j) \widehat{I}_{m,n,j;k} \right).$$

Insertion of the previous identity in (3.7) results in

$$S_M = Mh^{\frac{1}{2}} \int_{-\pi}^{\pi} \sum_{m,n=1}^{sd} \left| \frac{1}{M} \sum_{k=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} K_h(\omega - \omega_k) \left( -\frac{1}{s} \sum_{j=0}^{s-1} \Delta^{(1)}(j) \widehat{I}_{m,n,j;k} \right) \right|^2 d\omega. \quad (5.3)$$

The subsequent Lemmas 5.1 and 5.2 are concerned with the asymptotic behaviour of mean and variance of  $S_M$ , respectively. Finally, to complete the proof of Theorem 3.1, Lemma 5.3 deals with asymptotic normality of  $S_M$ .

**Lemma 5.1** (Computation of  $E(S_M)$ ).

Let  $s \geq 2$ . Suppose the assumptions (A), (K) and (B) are fulfilled and  $H_0$  is true. Then, it holds

$$E(S_M) = h^{-\frac{1}{2}} A_K \int_{-\pi}^{\pi} s \sum_{t=1}^{\lfloor \frac{s}{2} \rfloor} \kappa(t) |P(\mathbf{F}, t, \omega)|^2 d\omega + o(1),$$

where  $A_K$ ,  $\kappa(t)$  and  $P(\mathbf{F}, t, \omega)$  are defined before Theorem 3.1.

*Proof.*

Expanding the absolute value in (5.3), taking expectation and using the identity [cf. Brockwell and Davis (1991), p.444]

$$E(\widehat{I}_{mn}(\omega_{k_1}) \overline{\widehat{I}_{pq}(\omega_{k_2})}) = \text{Cov}(\widehat{I}_{mn}(\omega_{k_1}), \widehat{I}_{pq}(\omega_{k_2})) + g_{mn}(\omega_{k_1}) \overline{g_{pq}(\omega_{k_2})} + O\left(\frac{1}{M}\right)$$

results in

$$\begin{aligned} E(S_M) &= \frac{h^{\frac{1}{2}}}{M} \int_{-\pi}^{\pi} \sum_{m,n=1}^{sd} \sum_{k_1, k_2=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) \frac{1}{s^2} \sum_{j_1, j_2=0}^{s-1} \Delta^{(2)}(j_1, j_2) \\ &\quad \times \left( \text{Cov}(\widehat{I}_{m,n,j_1;k_1}, \widehat{I}_{m,n,j_2;k_2}) + g_{m,n,j_1,j_1;k_1} \overline{g_{m,n,j_2,j_2;k_2}} \right) d\omega + O(h^{\frac{1}{2}}) \\ &= A_1 + A_2 + o(1) \end{aligned}$$

with  $A_1$  and  $A_2$  corresponding to first and second term in parentheses above, respectively. The second term may be expressed as

$$A_2 = Mh^{\frac{1}{2}} \int_{-\pi}^{\pi} \sum_{m,n=1}^{sd} \left| \frac{1}{M} \sum_{k=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} K_h(\omega - \omega_k) \frac{1}{s} \sum_{j=0}^{s-1} \Delta^{(1)}(j) g_{m,n,j,j;k} \right|^2$$

and vanishes exactly, because of  $\sum_{j=0}^{s-1} \Delta^{(1)}(j) g_{m,n,j,j;k} = 0$  for all  $\omega \in [-\pi, \pi]$  due to the specific shape of  $\mathbf{g}(\omega)$  under the null of covariance stationarity of  $(\underline{Y}_t, t \in \mathbb{Z})$  as discussed in

Theorem 2.1. Now, it remains to consider  $A_1$ . Because of  $Cov(\widehat{I}_{mn}(\omega_{k_1}), \widehat{I}_{pq}(\omega_{k_2})) = O(\frac{1}{M})$  uniformly in  $\omega_{k_1}, \omega_{k_2}$  for  $|\omega_{k_1}| \neq |\omega_{k_2}|$  [cf. Hannan (1970), p.23] and the case  $\omega_{k_1} = -\omega_{k_2}$  does not make a contribution asymptotically, it suffices to consider the case  $\omega_{k_1} = \omega_{k_2}$  and  $Cov(\widehat{I}_{mn}(\omega_k), \widehat{I}_{pq}(\omega_k)) = g_{mp}(\omega_k)\overline{g_{nq}(\omega_k)} + g_{qm}(\omega_k)\overline{g_{pn}(\omega_k)}\delta_{k0} + O(\frac{1}{M})$  gives

$$\begin{aligned} A_1 &= \frac{h^{\frac{1}{2}}}{M} \int_{-\pi}^{\pi} \sum_{m,n=1}^{sd} \sum_{k=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} K_h^2(\omega - \omega_k) \frac{1}{s^2} \sum_{j_1, j_2=0}^{s-1} \Delta^{(2)}(j_1, j_2) g_{m,m,j_1,j_2;k} \overline{g_{n,n,j_1,j_2;k}} d\omega + o(1) \\ &= h^{-\frac{1}{2}} \int_{-\pi}^{\pi} \frac{1}{Mh} \sum_{k=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} K^2\left(\frac{\omega - \omega_k}{h}\right) \frac{1}{s^2} \sum_{j_1, j_2=0}^{s-1} \Delta^{(2)}(j_1, j_2) \left| \sum_{m=1}^{sd} g_{m,m,j_1,j_2;k} \right|^2 d\omega + o(1). \end{aligned}$$

Approximating the involved Riemann sum by its limiting integral and a standard substitution results in the asymptotically equivalent statistic

$$h^{-\frac{1}{2}} \frac{1}{2\pi} \int_{-\pi}^{\pi} K^2(v) dv \int_{-\pi}^{\pi} \frac{1}{s^2} \sum_{j_1, j_2=0}^{s-1} \Delta^{(2)}(j_1, j_2) \left| \sum_{m=1}^{sd} g_{m+m+j_1, m+j_2}(\omega) \right|^2 d\omega.$$

By manipulating the summation order and, once more, by exploiting the specific block entry-wise structure of  $\mathbf{g}(\omega)$  under  $H_0$ , the integrand of the second integral above may be expressed as

$$\frac{s(s-1)}{s^2} \left| \sum_{m=1}^{sd} g_{mm}(\omega) \right|^2 - \frac{2}{s} \sum_{t=1}^{\lfloor \frac{s-1}{2} \rfloor} \left| \sum_{m=1}^{sd} g_{m,m+dt}(\omega) \right|^2 - \frac{1}{s} \left| \sum_{m=1}^{sd} g_{m,m+d\frac{s}{2}}(\omega) \right|^2 \mathbf{1}(s \text{ even}). \quad (5.4)$$

Moreover, under  $H_0$ , it holds  $\sum_{m=1}^{sd} g_{m,m+dt}(\omega) = s \cdot \text{tr}(\mathbf{G}_{1,1+t}(\omega))$  and by using (2.8), the expression in (5.4) becomes

$$s(s-1) |\text{tr}(\mathbf{F}_{11}(\omega))|^2 - 2s \sum_{t=1}^{\lfloor \frac{s-1}{2} \rfloor} |\text{tr}(\mathbf{F}_{1,1+t}(\omega))|^2 - s \left| \text{tr}(\mathbf{F}_{1,1+\frac{s}{2}}(\omega)) \right|^2 \mathbf{1}(s \text{ even}),$$

which concludes the proof.  $\square$

**Lemma 5.2** (Computation of  $\text{Var}(S_M)$ ).

Let  $s \geq 2$ . Suppose the assumptions (A), (K) and (B) are fulfilled and  $H_0$  is true. Then, it holds

$$\text{Var}(S_M) = B_K \int_{-\pi}^{\pi} s \sum_{t=1}^{\lfloor \frac{s}{2} \rfloor} \kappa(t) \left| \sum_{n=1}^s Q(\mathbf{F}, n, t, \omega) \right|^2 d\omega + o(1),$$

where  $B_K$ ,  $\kappa(t)$  and  $Q(\mathbf{F}, n, t, \omega)$  are defined before Theorem 3.1.

*Proof.*

First, consider the second moment  $E(S_M^2)$  instead of  $\text{Var}(S_M)$ . Expanding all absolute values in (5.3) gives

$$\begin{aligned} & E(S_M^2) \tag{5.5} \\ &= \frac{h}{M^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{m_1, n_1, m_2, n_2} \sum_{k_1, k_2, k_3, k_4} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) K_h(\lambda - \omega_{k_3}) K_h(\lambda - \omega_{k_4}) \\ &\quad \times \frac{1}{s^4} \sum_{j_1, j_2, j_3, j_4=0}^{s-1} \Delta^{(4)}(j_1, j_2, j_3, j_4) E(\widehat{I}_{m_1, n_1, j_1; k_1} \overline{\widehat{I}_{m_1, n_1, j_2; k_2}} \widehat{I}_{m_2, n_2, j_3; k_3} \overline{\widehat{I}_{m_2, n_2, j_4; k_4}}) d\omega d\lambda. \end{aligned}$$

Due to  $\widehat{I}_{m+dj,n+dj}(\omega_k) = I_{m+dj,n+dj}(\omega_k)e^{-i\lfloor \frac{m-n}{d} \rfloor \frac{1}{s}\omega_k}$  and after plugging-in for the periodogram and using  $X_{t,m+dj} = \sum_{\nu=-\infty}^{\infty} \sum_{\mu=1}^{sd} B_{\nu;m+dj,\mu} e_{t-\nu,\mu}$ , essentially, one has to evaluate

$$E(e_{t_1-\nu_1,\mu_1} e_{t_2-\nu_2,\mu_2} e_{t_3-\nu_3,\mu_3} e_{t_4-\nu_4,\mu_4} e_{t_5-\nu_5,\mu_5} e_{t_6-\nu_6,\mu_6} e_{t_7-\nu_7,\mu_7} e_{t_8-\nu_8,\mu_8})$$

to compute the expectation on the right-hand side of (5.5). Taking (5.5) into account, the cases with largest contribution are those consisting of four different pairs of  $e_t$ 's and there are exactly five combinations that do not vanish and make contributions asymptotically to  $E(S_M^2)$ .

The first relevant case is  $t_1 - \nu_1 = t_3 - \nu_3$ ,  $t_2 - \nu_2 = t_4 - \nu_4$ ,  $t_5 - \nu_5 = t_7 - \nu_7$  and  $t_6 - \nu_6 = t_8 - \nu_8$ . This combination implies  $\omega_{k_1} = \omega_{k_2}$  and  $\omega_{k_3} = \omega_{k_4}$  (in the limit) and in this situation both integrals with respect to  $\omega$  and  $\lambda$  in (5.5) separate and its contribution to  $E(S_M^2)$  cancels out with  $-(E(S_M))^2$  asymptotically when evaluating  $Var(S_M)$ .

All of the other four relevant combinations of index pairs converge to the same limit and as a representative consider  $t_1 - \nu_1 = t_6 - \nu_6$ ,  $t_2 - \nu_2 = t_5 - \nu_5$ ,  $t_3 - \nu_3 = t_8 - \nu_8$  and  $t_4 - \nu_4 = t_7 - \nu_7$ . In this case, the expectation on the right-hand side in (5.5) is (asymptotically) equal to

$$g_{m_1,n_2,j_1,j_3;k_1} \overline{g_{n_1,m_2,j_1,j_3;k_1} g_{m_1,n_2,j_2,j_4;k_2} g_{n_1,m_2,j_2,j_4;k_2} \delta_{k_1,k_3} \delta_{k_2,k_4}}$$

Now, inserting this term in (5.5), taking all four relevant combinations into considerations which gives a factor 4 and further calculations yield

$$\begin{aligned} Var(S_M) &= \frac{4h}{M^2} \sum_{k_1, k_2 = -\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} \left( \int_{-\pi}^{\pi} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) d\omega \right)^2 \\ &\quad \times \frac{1}{s^4} \sum_{j_1, j_2, j_3, j_4 = 0}^{s-1} \Delta^{(4)}(j_1, j_2, j_3, j_4) \left| \sum_{m,n=1}^{sd} g_{m,n,j_1,j_3;k_1} \overline{g_{m,n,j_2,j_4;k_2}} \right|^2 + o(1) \quad (5.6) \\ &= \frac{4h}{M^2} \sum_{k_1, k_2 = -\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} \left( \int_{-\pi}^{\pi} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) d\omega \right)^2 R(\omega_{k_1}, \omega_{k_2}) + o(1) \end{aligned}$$

with an obvious notation for  $R(\omega_{k_1}, \omega_{k_2})$ . Approximation of both Riemann sums by their limiting integrals and standard substitutions yield the asymptotically equivalent expression

$$\frac{1}{\pi^2} \int_{-2\pi}^{2\pi} \left( \int_{-\pi}^{\pi} K(x) K(x+z) dx \right)^2 dz \int_{-\pi}^{\pi} R(\omega, \omega) d\omega.$$

To get rid of (most of) the redundancy contained in the sums over  $j_1, j_2, j_3$  and  $j_4$  in  $R(\omega) = R(\omega, \omega)$  as defined above in (5.6), consider all nine combinations of the Cartesian product

$$\{\{j_1 = j_3\}, \{j_1 < j_3\}, \{j_1 > j_3\}\} \times \{\{j_2 = j_4\}, \{j_2 < j_4\}, \{j_2 > j_4\}\}. \quad (5.7)$$

For instance, due to the specific shape of  $\mathbf{g}(\omega)$  under  $H_0$  as shown in Theorem 2.1, the identity  $\sum_{m,n=1}^{sd} |g_{mn}(\omega)|^2 = s \sum_{n=1}^s \text{tr}(\mathbf{G}_{1,n}(\omega) \mathbf{G}_{1,n}^H(\omega))$  and (2.8), the first case  $\{j_1 = j_3\} \times \{j_2 = j_4\}$  makes a contribution of

$$\begin{aligned} &\frac{1}{s^4} \sum_{j_1, j_2 = 0}^{s-1} \left( \Delta^{(1)}(j_1) \Delta^{(1)}(j_2) \right)^2 \left| \sum_{m,n=1}^{sd} g_{m+dj_1, n+dj_1}(\omega) \overline{g_{m+dj_2, n+dj_2}(\omega)} \right|^2 \\ &= (s-1)^2 \left| \sum_{n=1}^s \text{tr}(\mathbf{F}_{1,n}(\omega) \mathbf{F}_{1,n}^H(\omega)) \right|^2 \end{aligned}$$

to the asymptotic variance of  $S_M$ . Similar results hold for all other combinations in (5.7) and, eventually, lengthy and tedious calculations and repeated applications of Theorem 2.1 give

$$\begin{aligned} R(\omega) &= s(s-1) \left| \sum_{n=1}^s \text{tr} \left( \mathbf{F}_{1,n}(\omega) \mathbf{F}_{1,n}^H(\omega) \right) \right|^2 - 2s \sum_{t=1}^{\lfloor \frac{s-1}{2} \rfloor} \left| \sum_{n=1}^s \text{tr} \left( \mathbf{F}_{1,n}(\omega) \mathbf{F}_{1,n+t}^H(\omega) \right) \right|^2 \\ &\quad - s \left| \sum_{n=1}^s \text{tr} \left( \mathbf{F}_{1,n}(\omega) \mathbf{F}_{1,n+\frac{s}{2}}^H(\omega) \right) \right|^2 \mathbf{1}(s \text{ even}), \end{aligned}$$

which completes the proof.  $\square$

**Lemma 5.3** (Asymptotic normality of  $S_M$ ).

Let  $s \geq 2$ . Suppose the assumptions (A), (K) and (B) are fulfilled and  $H_0$  is true. Then, it holds that

$$S_M - E(S_M)$$

is asymptotically normally distributed.

*Proof.*

Due to (3.3), entries of the modified periodogram matrix may be expressed as

$$\widehat{I}_{m+dj, n+dj}(\omega_k) = \left( \frac{1}{2\pi} \sum_{l=-(M-1)}^{M-1} \frac{1}{M} \sum_{t=\max(1,1+l)}^{\min(M,M+l)} X_{t,m+dj} X_{t-l,n+dj} e^{-il\omega_k} \right) \left( e^{-i \lfloor \frac{m-n}{d} \rfloor \frac{\omega_k}{s}} \right).$$

Inserting this identity in (5.3) and a change of summation order yields

$$\begin{aligned} S_M &= \frac{h^{\frac{1}{2}}}{M} \int_{-\pi}^{\pi} \sum_{m,n=1}^{sd} \left| \frac{1}{2\pi} \sum_{l=-(M-1)}^{M-1} \left( \frac{1}{Mh} \sum_{k=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} K \left( \frac{\omega - \omega_k}{h} \right) e^{-il\omega_k} e^{-i \lfloor \frac{m-n}{d} \rfloor \frac{\omega_k}{s}} \right) \right. \\ &\quad \left. \times \left( -\frac{1}{s} \sum_{j=0}^{s-1} \Delta^{(1)}(j) \sum_{t=\max(1,1+l)}^{\min(M,M+l)} X_{t,m+dj} X_{t-l,n+dj} \right) \right|^2 d\omega. \end{aligned}$$

The asymptotically equivalent statistic

$$\begin{aligned} &\frac{h^{\frac{1}{2}}}{M} \int_{-\pi}^{\pi} \sum_{m,n=1}^{sd} \left| \frac{1}{2\pi} \sum_{l=-(M-1)}^{M-1} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} K(v) e^{-il(\omega-hv)} e^{-i \lfloor \frac{m-n}{d} \rfloor \frac{\omega-hv}{s}} dv \right) \right. \\ &\quad \left. \times \left( -\frac{1}{s} \sum_{j=0}^{s-1} \Delta^{(1)}(j) \sum_{t=\max(1,1+l)}^{\min(M,M+l)} X_{t,m+dj} X_{t-l,n+dj} \right) \right|^2 d\omega \end{aligned}$$

that is obtained by approximating the involved Riemann sum by its limiting integral and by standard substitution, can be simplified by using  $|e^{-i \lfloor \frac{m-n}{d} \rfloor \frac{\omega}{s}}| = 1$ ,  $|e^{i \lfloor \frac{m-n}{d} \rfloor \frac{hv}{s}} - 1| = O(h)$  and by defining  $k(l, h) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(v) \cos(lhv) dv$  and

$$a_l = k(l, h) \left( -\frac{1}{s} \sum_{j=0}^{s-1} \Delta^{(1)}(j) \sum_{t=\max(1,1+l)}^{\min(M,M+l)} X_{t,m+dj} X_{t-l,n+dj} \right). \quad (5.8)$$

Altogether, this results in the following representation of  $S_M$  and an application of Parseval's identity gives

$$\begin{aligned} S_M &= \frac{h^{\frac{1}{2}}}{2\pi M} \sum_{m,n=1}^{sd} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{l=-(M-1)}^{M-1} a_l e^{-il\omega} \right|^2 d\omega + o_P(1) \\ &= \frac{h^{\frac{1}{2}}}{2\pi M} \sum_{m,n=1}^{sd} \sum_{l=-(M-1)}^{M-1} a_l^2 + o_P(1). \end{aligned}$$

Now, insertion of (5.8) in the last expression on the right-hand side above and some summation manipulations yield

$$\begin{aligned} &S_M \\ &= \frac{h^{\frac{1}{2}}}{2\pi M s^2} \sum_{j_1, j_2=0}^{s-1} \Delta^{(2)}(j_1, j_2) \\ &\quad \times \sum_{l=-(M-1)}^{M-1} k^2(l, h) \sum_{t_1, t_2=\max(1, 1+l)}^{\min(M, M+l)} \left( \sum_{m=1}^{sd} X_{t_1, m+dj_1} X_{t_2, m+dj_2} \right) \left( \sum_{n=1}^{sd} X_{t_1-l, n+dj_1} X_{t_2-l, n+dj_2} \right) \\ &= \frac{h^{\frac{1}{2}}}{2\pi M s^2} \sum_{j_1, j_2=0}^{s-1} \Delta^{(2)}(j_1, j_2) \\ &\quad \times \sum_{t_1, t_2=1}^M \left( \sum_{l=1-\min(t_1, t_2)}^{M-\max(t_1, t_2)} k^2(l, h) \left( \sum_{n=1}^{sd} X_{t_1+l, n+dj_1} X_{t_2+l, n+dj_2} \right) \right) \left( \sum_{m=1}^{sd} X_{t_1, m+dj_1} X_{t_2, m+dj_2} \right) \end{aligned}$$

The last right-hand side above can be split up into cases  $t_1 = t_2$  and  $t_1 \neq t_2$ . The first case vanishes asymptotically in  $S_M - E(S_M)$  with  $E(S_M)$  derived in Lemma 5.1 and the remaining second case can be treated with Theorem 2.1 in Gao and Hong (2008) and the claimed asymptotic normality follows consequently from Assumption (A).  $\square$

### Proof of Theorem 3.2.

By using identity (5.3) and due to

$$\frac{1}{M} \sum_{k=-\lfloor \frac{M-1}{2} \rfloor}^{\lfloor \frac{M}{2} \rfloor} K_h(\omega - \omega_k) \left( -\frac{1}{s} \sum_{j=0}^{s-1} \Delta^{(1)}(j) \widehat{I}_{m,n,j;k} \right) = -\frac{1}{s} \sum_{j=0}^{s-1} \Delta^{(1)}(j) g_{m+dj, n+dj}(\omega) + o_P(1)$$

uniformly in  $\omega$ , we get

$$\begin{aligned} M^{-1} h^{-\frac{1}{2}} S_M &= \int_{-\pi}^{\pi} \sum_{m,n=1}^{sd} \left| -\frac{1}{s} \sum_{j=0}^{s-1} \Delta^{(1)}(j) g_{m+dj, n+dj}(\omega) \right|^2 d\omega + o_P(1) \\ &= \int_{-\pi}^{\pi} \|\mathbf{g}(\omega) - \mathbf{g}_1(\omega)\|^2 d\omega + o_P(1), \end{aligned}$$

which completes the proof.  $\square$

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