A NOTE ON USING PERIODOGRAM-BASED DISTANCES FOR COMPARING SPECTRAL DENSITIES

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Abstract. Motivated by a recent paper of Caiado et. al. (2009), we investigate testing problems for spectral densities of time series with unequal sample sizes. We thereby focus on analyzing their mathematical properties and illustrate our results in a small simulation study

1. Motivation

Various authors have studied the testing problem for equality of spectral densitites of two (or more) different time series. For nonparametric situations most of the proposed procedures are only reasoned by simulations or/and heuristic proofs, see e.g. Coates and Diggle (1986), Pötscher and Reschenhofer (1988), Diggle and Fisher (1991) and Maharaj (2002). Two exceptions from this custom are given by the recently published papers by Eichler (2008) and Dette and Paparoditis (2009). In these articles the mathematical properties of the proposed tests based on the smoothed periodograms as estimators for the spectra are also analyzed in detail.
However, all of the above mentioned citations have in common that their procedures can only be applied in settings with equal sample size. Since this non-existence of statistical tools for comparing time series with unbalanced samples also occurs in other statistical situations, Caiado et. al. (2009) have proposed the usage of periodogram-based distances. In the sequel we will act on their suggestions by using (some versions of) their distances as test statistics for the above testing problem. Moreover, we will analyze the asymptotic properties of the corresponding tests.

2. Notation and assumptions

Consider two independent weakly stationary processes $\mathbf{X} := (X_t, t \in \mathbb{Z})$ and $\mathbf{Y} := (Y_t, t \in \mathbb{Z})$ with $E(X_t) = E(Y_t) = 0$ and existing and absolutely summable autocovariance functions γ_X and γ_Y, respectively. In this case their spectral densities

$$f_X(\omega) := \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma_X(h) \exp(-ih\omega), \qquad -\pi < \omega < \pi, \tag{2.1}$$

and f_Y exist. Note that f_X and f_Y are even functions, so that we only need to consider the case $\omega > 0$. Based on observed samples $X_1, \dots, X_{n_1}$ and $Y_1, \dots, Y_{n_2}$ with $n_1 \geq n_2$, it is well known that their corresponding periodograms

$$I_{n_1,X}(\omega) := \frac{1}{2\pi n_1} \left| \sum_{t=1}^{n_1} X_t \exp(-it\omega) \right|^2 \tag{2.2}$$

and $I_{n_2,Y}(\omega)$ are uniformly mean-consistent estimators for f_X and f_Y respectively. In order to compare f_X and f_Y, Caiado et. al. (2006, 2009) have, among other things, suggested to use

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different periodogram-based distances. If you look at the test statistic of classical goodness-of-fit tests like the Cramér-Von Mises, Anderson-Darling or Kolmogorov-Smirnov test (where the empirical c.d.f. are used as estimators for the correct c.d.f. and inserted in some kind of (pseudo-)distance), this approach appears to be very natural.
If the sample sizes n_1, n_2 are not equal, say $n_1 > n_2$, the sets of the corresponding Fourier-frequencies are not equal as well and their intersection is in general very small or even empty. Hence it arises the question at which common frequencies the two periodograms should be compared. As an answer Caiado et. al. (2009) have suggested three different approaches to overcome this problem. First they propose to interpolate the periodograms to make them better comparable. This approach can only be reasoned by an (at least approximately) linear behavior of the periodograms $\omega \mapsto I(\omega)$, which is in general not true. Their second suggestion is to fill up the smaller time series with $n_1 - n_2$ zero's. However, this distorts the structure of the smaller series and is therefore at most suboptimal. In their third proposal they evaluate the longer series $X_1, \dots, X_{n_1}$ at the Fourier frequencies of $Y_1, \dots, Y_{n_2}$. This approach is pretty canonical but has the disadvantage that it does not use the whole information about $\mathbf{X}$ contained in the longer data stretch.
More general than in the last case we will compare the periodograms at common frequencies $(\omega_j)_j = (\omega_j(n))_j$ that differ from the standard Fourier-frequencies $\{2\pi j/n_1 : j = 1, \dots, n_1\}$ and $\{2\pi j/n_2 : j = 1, \dots, n_2\}$ in Section 3. Two possible statistics are given by

$$T_n^{(1)} = \frac{1}{n_2} \sum_{j=1}^{n_2} \big(I_{n_1,X}(\omega_j) - I_{n_2,Y}(\omega_j)\big)^2 \tag{2.3}$$

and

$$T_n^{(2)} = \frac{1}{n_2} \sum_{j=1}^{n_2} \log^2 \left(\frac{I_{n_1,X}(\omega_j)}{I_{n_2,Y}(\omega_j)}\right) \mathbf{1}_{\{I_{n_2,Y}(\omega_j) I_{n_1,X}(\omega_j) \neq 0\}}, \tag{2.4}$$

see Caiado et.al. (2009) (see definitions (4) and (7) without standardization and interpolation). Here we use the convention that a summand in (2.4) is zero if the corresponding indicator function is zero. In comparison to the simulation-based reasoning of Caiado et. al. (2006) and (2009) we will analyze versions of these statistics from a mathematical point of view.

3. Main results

Throughout we will assume that $\min(n_1, n_2) \to \infty$ and that $D := \{\omega_1, \omega_2, \dots\}$ is a dense subset of $\mathbb{Q} \cap (0, \pi)$ sucht that $\omega_j \neq \omega_k$ for all $k \neq j$. For example choose $D := \cup_n B_n$ with $B_n := A_n \cap_{k=1}^{n-1} A_k^c$ for $A_n := \cup_{k=1}^{2^n - 1} \{\frac{k\pi}{2^n}\}, A_0 := \emptyset$, with ordering, i.e. $\omega_1 = \pi/2, \omega_2 = \pi/4, \omega_3 = 3\pi/4, \dots$. Recall that it suffices to evaluate the periodograms only on $(0, \pi)$ for symmetry reasons. In the sequel we consider the hypothesis

$$H_0 : \{f_X(\omega) = f_Y(\omega) \text{ for all } \omega \in (0, \pi)\}. \tag{3.1}$$

For $m := \lfloor n_2/2 \rfloor$ (or even more general choices of $m = m_n$ with $m_n \to \infty$ for $n \to \infty$) and a positive and summable sequence $(c_j)_j \in \ell_1$ (e.g. we may choose $c_j = j^{-s}$ with $s > 1$) that does not depend on n_i, $i = 1, 2$, we will now consider slightly modified versions of (2.3)

$$T_n^{(3)} := \sum_{j=1}^{m} c_j \big(I_{n_1,X}(\omega_j) - I_{n_2,Y}(\omega_j)\big)^2 \tag{3.2}$$

and (2.4)

$$T_n^{(4)} := \sum_{j=1}^{m} c_j \log^2\Big(\frac{I_{n_1,X}(\omega_j)}{I_{n_2,Y}(\omega_j)}\Big)\mathbf{1}_{\{I_{n_2,Y}(\omega_j)I_{n_1,X}(\omega_j)\neq 0\}}, \tag{3.3}$$

where we again use the convention that a summand in (3.3) is zero if the indicator function is zero. Remark that, in contrast to (2.3) and (2.4), the summable condition $(c_j)_j \in \ell_1$ is needed to gain a non-degenerated limit distribution of the test statistic. For $T_n^{(3)}$ this can be seen in the following Theorem.

Theorem 3.1. *Suppose that* $\mathbf{X}$ *and* $\mathbf{Y}$ *are either linear processes with finite fourth moments or fulfill Assumption 2.6.1. of Brillinger (1981). Then, under* H_0*, we have convergence in distribution*

$$T_n^{(3)} \xrightarrow{\mathcal{D}} \sum_{j=1}^{\infty} c_j f_X^2(\omega_j) D_j^2 =: S^{(3)}, \tag{3.4}$$

where $(D_n)_{n\in\mathbb{N}}$ *is a sequence of independent, double exponentially distributed r.v.s with p.d.f.* $g(x) = \exp(-|x|)/2$*. Moreover, the limit* $S^{(3)}$ *converges a.s. and in* L_1*.*

Proof. We will apply Theorem 4.2. from Billingsley (1968). Remark first that for every fixed $k \in \mathbb{N}$ we have joint convergence in distribution

$$\begin{aligned}&\big(I_{n_1,X}(\omega_1), I_{n_1,X}(\omega_2), \ldots, I_{n_1,X}(\omega_k), I_{n_2,Y}(\omega_1), I_{n_2,Y}(\omega_2), \ldots, I_{n_2,Y}(\omega_k)\big)\\ &\xrightarrow{\mathcal{D}} \big(f_X(\omega_1)E_1, f_X(\omega_2)E_2, \ldots, f_X(\omega_k)E_k, f_Y(\omega_1)\hat{E}_1, f_Y(\omega_2)\hat{E}_2, \ldots, f_Y(\omega_k)\hat{E}_k\big),\end{aligned}$$

where $E_i, \hat{E}_i$ are i.i.d. standard exponential distributed r.v.s., see Theorem 10.3.2. in Brockwell and Davis (1991) for the case of a linear process and Theorem 5.2.6. in Brillinger (1981) for the second case. Since the r.v.s

$$D_i := E_i - \hat{E}_i, \quad i \in \mathbb{N},$$

are i.i.d. with p.d.f. g expectation $E(D_1) = 0$ and variance $Var(D_1) = 2$ we obtain

$$T_{n,k}^{(3)} := \sum_{j=1}^{k} c_j \big(I_{n_1,X}(\omega_j) - I_{n_2,Y}(\omega_j)\big)^2 \xrightarrow{\mathcal{D}} \sum_{j=1}^{k} c_j f_X^2(\omega_j) D_j^2 =: S_k^{(3)}.$$

Moreover, the limit is a Cauchy-sequence in L_1 since

$$E(|S_{k_1}^{(3)} - S_{k_2}^{(3)}|) = \sum_{j=k_2+1}^{k_1} c_j f_X^2(\omega_j) Var(D_j) = 2 \sum_{j=k_2+1}^{k_1} c_j f_X^2(\omega_j) \le \epsilon$$

for all fixed $\epsilon > 0$ and all sufficiently large $k_2 \le k_1$. Hence we have

$$\sum_{j=1}^{k} c_j f_X^2(\omega_j) D_j^2 \longrightarrow \sum_{j=1}^{\infty} c_j f_X^2(\omega_j) D_j^2 \qquad \text{a.s. and in } L_1$$

as $k \to \infty$. Thus it remains to treat the remainder term $T_n^{(3)} - T_{n,k}^{(3)}$:

$$\begin{aligned} E(|T_n^{(3)} - T_{n,k}^{(3)}|) &= \sum_{j=k+1}^{m} c_j E\big((I_{n_1,X}(\omega_j) - I_{n_2,Y}(\omega_j))^2\big)\\ &= \sum_{j=k+1}^{m} c_j \big(2Var(I_{n_1,X}(\omega_j)) + Cov(I_{n_1,X}(\omega_j), I_{n_1,Y}(\omega_j)) + o(1)\big)\\ &= \sum_{j=k+1}^{m} c_j (2f_X^2(\omega_j) + o(1)). \end{aligned}$$

Since $f_X(\omega)$ is uniformly bounded by assumption and $(c_j)_j$ is absoulutely summable, this shows

$$\lim_{k\to\infty}\limsup_{n\to\infty} P(|T_n^{(3)} - T_{n,k}^{(3)}| \geq \epsilon) = 0 \tag{3.5}$$

for all $\epsilon > 0$ and completes the proof. □

It is interesting to note that the limit looks similar to limits of classical goodness-of fit tests. Take for example the one-sample Cramér-Von Mises test statistic $W_n^2 := n\int(F_n - F_0)^2 dF_0$ for the null hypothesis $\{F = F_0\}$, where F_n is the empirical c.d.f. of i.i.d. r.v. with c.d.f. F. Here W_n^2 converges in distribution to $W^2 := \sum_{j=1}^{\infty}(j\pi)^{-2}N_j^2$ under the null, where $N_1, N_2, \dots$ is an i.i.d. sequence of standard normal random variables, see e.g. chapter 5 in Shorack and Wellner (1986), where you can find many other examples as well.
However, remark that the limit of $T_n^{(3)}$ under H_0 depends on unknown values of the spectra. Hence the above results cannot be applied directly for statistical inference. As can be seen in the proof above, this problem may be solved by using a statistic based on ratios of the two periodograms. This leads to analyse the asymptotic distribution of $T_n^{(4)}$. Before we start this investigation note that a version of $T_n^{(4)}$ without the logarithm would not converge as above, because under H_0 the ratios of the periodograms converge in distribution to a shifted Pareto-$P(1,1)$ distribution, of whose first moment does not even exist.

Theorem 3.2. *Suppose that the conditions of Theorem 3.1 are satisfied. Moreover, assume that*

$$\inf_{\substack{n_1\geq n_0\\ 0<\omega<\pi}} I_{n_1,X}(\omega) > 0 \quad \textit{and} \quad \inf_{\substack{n_2\geq n_0\\ 0<\omega<\pi}} I_{n_2,Y}(\omega) > 0 \tag{3.6}$$

hold almost surely for some $n_0 \in \mathbb{N}$ large enough. Then we have convergence in distribution under H_0

$$T_n^{(4)} \xrightarrow{\mathcal{D}} \sum_{j=1}^{\infty} c_j Z_j^2 =: S^{(4)}, \tag{3.7}$$

where $(Z_n)_{n\in\mathbb{N}}$ is a sequence of independent, standard logistic distributed r.v.s with c.d.f. $F(x) = 1/(\exp(-x)+1)$. Moreover, the limit $S^{(4)}$ converges a.s. and in L_1.

Proof. Remark that the r.v.s $Z_i := -\log\Big(\frac{E_i}{\hat{E}_i}\Big)$, $i \in \mathbb{N}$, are i.i.d. with c.d.f. F, expectation $E(Z_1) = 0$ and variance $Var(Z_1) = \pi^2/3$. Hence we can obtain as above

$$T_{n,k}^{(4)} := \sum_{j=1}^{k} c_j \log^2\Big(\frac{I_{n_1,X}(\omega_j)}{I_{n_2,Y}(\omega_j)}\Big)\mathbf{1}_{\{I_{n_1,X}(\omega_j)I_{n_2,Y}(\omega_j)\neq 0\}} \xrightarrow{\mathcal{D}} \sum_{j=1}^{k} c_j Z_j^2 =: S_k^{(4)},$$

where the limit is a Cauchy-sequence in L_1. By using the inequality $\log(I_{n_1,X}(\omega)) \leq I_{n_1,X}(\omega)$ for $I_{n_1,X}(\omega) \geq 1$ and assumption (3.6) for $I_{n_1,X}(\omega) < 1$ straightforward calculations show that the remainder term $T_n^{(4)} - T_{n,k}^{(4)}$ is negligible. Hence another application of Theorem 4.2. from Billingsley (1968) completes the proof. □

Note that (3.6) appears to be quite restrictive, but for periodograms based on observations from a purely non-deterministic process they equal zero with probability zero anyway. Therefore, condition (3.6) is just slightly stronger.
Remark further that for practical purposes it is convenient to approximate the distribution of $S^{(4)}$ by the law of $S_K = \sum_{j=1}^{K} c_j Z_j^2$ for large K.

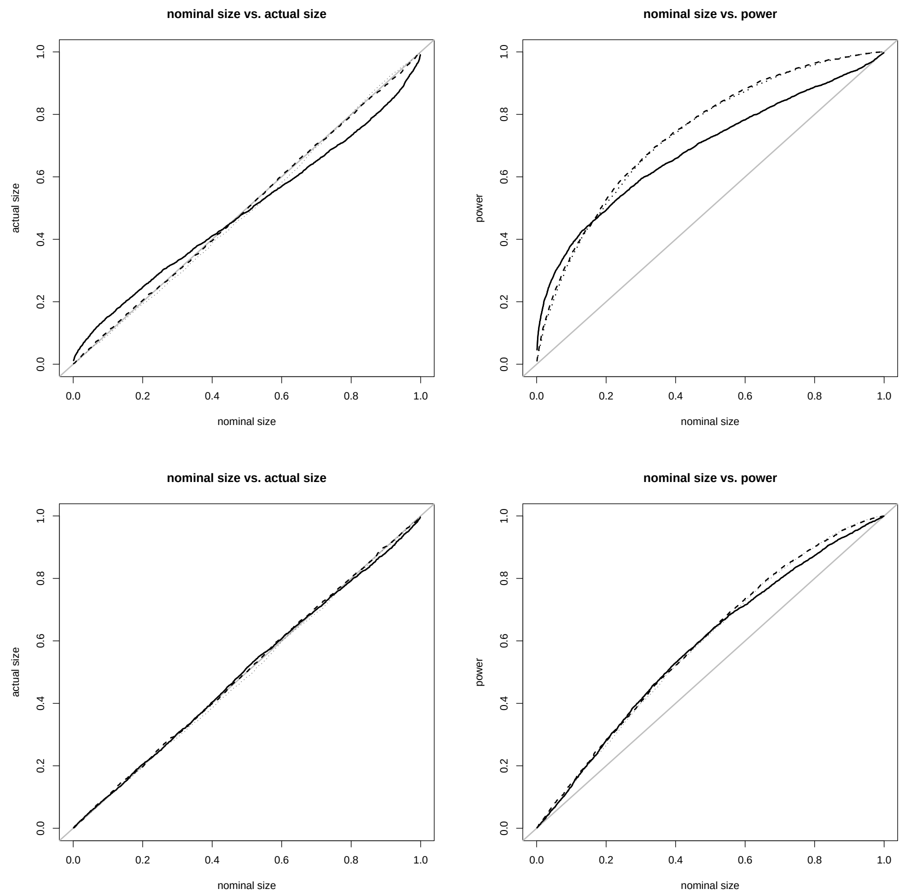

FIGURE 1. P-value plots (left) and size power curves (right) for testing equality of two spectral densities. In all panels, $T = 5000$ univariate time series X and Y coming from models (Ia) and (Ib) have been generated and $T_n^{(4)}$ has been executed for weights c_j (first row) and d_j (second row) for sample sizes $(n_1, n_2) = (50, 75)$ (solid line), $(n_1, n_2) = (200, 300)$ (dashed line) and $(n_1, n_2) = (800, 1200)$ (dashed and dotted line).

		H_0			H_1		
α \ n_1/n_2		50/75	200/300	800/1200	50/75	200/300	800/1200
0.01	c_j	0.0360	0.0106	0.0096	0.1400	0.0686	0.0588
	d_j	0.0096	0.0082	0.0110	0.0124	0.0188	0.0126
0.05	c_j	0.0968	0.0522	0.0512	0.2874	0.2280	0.2104
	d_j	0.0544	0.0520	0.0492	0.0656	0.0780	0.0672
0.10	c_j	0.1520	0.1038	0.0922	0.3828	0.3516	0.3394
	d_j	0.1062	0.1006	0.1002	0.1336	0.1434	0.139

TABLE 1. Nominal size vs. actual size and power for models (Ia) and (Ib).

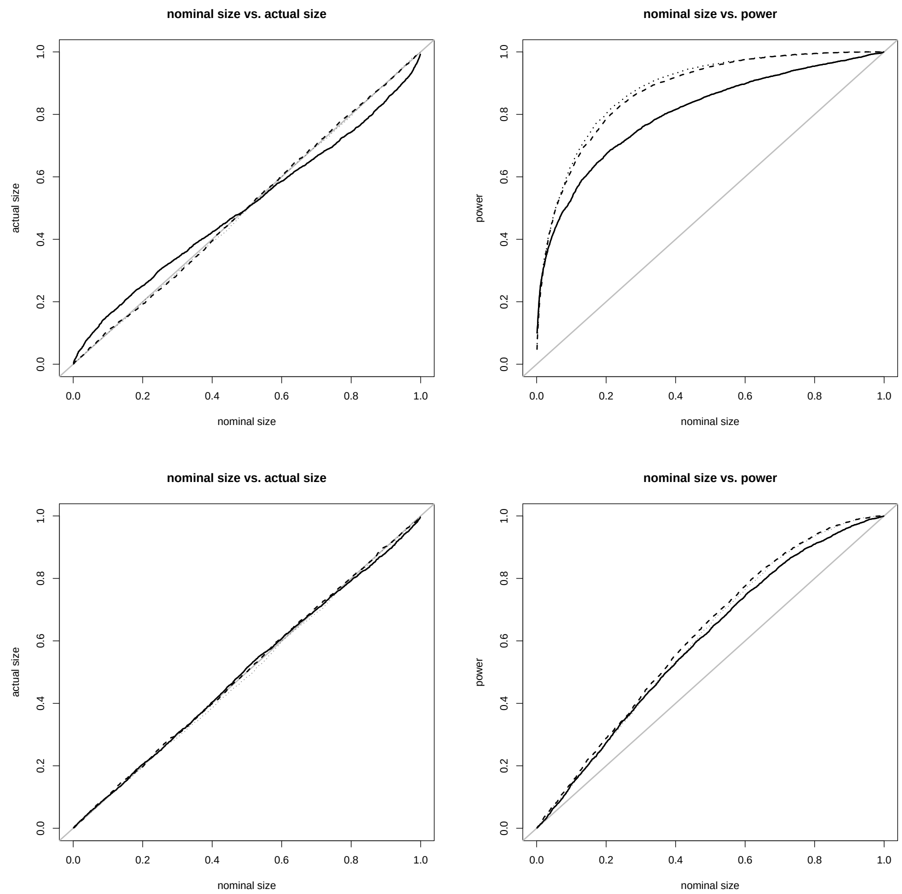

Figure 2. P-value plots (left) and size power curves (right) for testing equality of two spectral densities. In all panels, $T = 5000$ univariate time series X and Y coming from models (IIa) and (IIb) have been generated and $T_n^{(4)}$ has been executed for weights c_j (first row) and d_j (second row) for sample sizes $(n_1, n_2) = (50, 75)$ (solid line), $(n_1, n_2) = (200, 300)$ (dashed line) and $(n_1, n_2) = (800, 1200)$ (dashed and dotted line).

α \ n_1/n_2		H_0 50/75	H_0 200/300	H_0 800/1200	H_1 50/75	H_1 200/300	H_1 800/1200
0.01	c_j	0.0262	0.0120	0.0120	0.2422	0.2156	0.2388
	d_j	0.0112	0.0096	0.0122	0.0118	0.0124	0.0142
0.05	c_j	0.0914	0.0540	0.0522	0.4252	0.4776	0.4800
	d_j	0.0554	0.0536	0.0504	0.0664	0.0728	0.0742
0.10	c_j	0.1552	0.1066	0.0998	0.5294	0.6192	0.6370
	d_j	0.1022	0.1036	0.1002	0.141	0.1448	0.1358

Table 2. Nominal size vs. actual size and power for models (IIa) and (IIb).

4. Simulations

To illustrate the theoretically derived results, we have tested the null of equal spectral densities of two processes $(X_t, t \in \mathbb{Z})$ and $(Y_t, t \in \mathbb{Z})$ which are partially observed with (different) sample sizes n_1 and n_2, respectively, in two situations. We have chosen to investigate the small sample behaviour of $T_n^{(4)}$ rather than $T_n^{(3)}$ because contrary to $T_n^{(3)}$ the distribution of $T_n^{(4)}$ does not depend on unknown quantities under the null.

We investigate the behaviour of the test under the null of equal spectral densities and the behaviour under certain alternatives for two different autoregressive models and two different choices of weights.
In a first model (Ia), we consider independent $AR(1)$ processes (X_t) and (Y_t) with parameter $a = 0.5$ and standard normally distributed white noise under the null [cf. left panels of Figure 1], i.e. $X_t = aX_{t-1} + e_t$ and $Y_t = aY_{t-1} + \epsilon_t$.
To analyze the size, we have used again $AR(1)$ processes with $a = 0.5$, but the normally distributed white noise process used for (X_t) has unit variance and the variance of the white noise for (Y_t) is twice as large. Note that in this situation, denoted as model (Ib), the spectral densities of (X_t) devided by those of (Y_t) equals 2 on $(0, \pi)$.
We have done the simulations for sample sizes $(n_1, n_2) = (50, 75)$, $(n_1, n_2) = (200, 300)$ and $(n_1, n_2) = (800, 1200)$ and for two sets of weights (c_j) and (d_j) corresponding two the first and second row of panels in Figure 1, respectively. More precisely, we have used weights of finite support $c_j = \frac{1}{2^7-1}$ for $j = 1, \ldots, 2^7 - 1$ and $c_j = 0$ otherwise for the first row of panels in Figure 1. For the second row of panels in Figure 1 we have utilized $d_j = \frac{1}{j^{1.2}} \cdot \frac{1}{2^{j-1}}$ for the jth stage of dyadic numbers in $(0, \pi)$, i.e. the frequency $\omega_1 = \frac{\pi}{2}$ gets the weight d_1, the frequencies $\omega_2 = \frac{\pi}{4}$ and $\omega_3 = \frac{3\pi}{4}$ get the weight d_2 and so on. Observe that the weights have to be absolutely summable and are not allowed to depend on the sample sizes n_1 and n_2 within the framework of this paper. Moreover, notice that the first weights asses the first $2^7 - 1$ dyadic frequencies equally, whereas the second put different weight on frequencies from different sets $B_j := A_j \cap_{k=1}^{j-1} A_k^c$ with $A_j := \cup_{k=1}^{2^j-1} \{\frac{k\pi}{2^j}\}, A_0 := \emptyset$.
In all situations, we have computed the test statistic at the dyadic numbers of the interval $(0, \pi)$ up to $K_{dya} = 2^{10}$. To calculate the limiting distribution under the null, we have generated $K_{rv} = 2^{10}$ standard logistic distributed random numbers, computed their square and used the weights (c_j) and (d_j) to eventually obtain one realization of the test statistic under the null. This has been executed $K_{loop} = 10000$ times to approximate sufficiently well the true distribution and their empirical quantiles which are finally used as critical values.

The results can be found in Figure 1 and Table 1. There the two panels from the first column of Figure 1 illustrate the theoretical result of Theorem 3.2 that the distribution of the test statistic $T_n^{(4)}$ can be approximated under the null very well by the law of $S^{(4)}$. This goes hand in hand with the convergence of the actual size to the given error probability α of first kind under H_0, see Table 1. However, the behaviour under the alternative seems to be pretty poor. Indeed, the results from Table 1 and the second column from Figure 1 suggest that the tests are not consistent under the alternative for both choices of weights. A theoretical explanation for this behaviour is given in the next section. There we show that tests based on (3.3) are in general not consistent under general alternatives for any choices of weights that does not depend on n_i and are therefore not preferable to test H_0.

In a second situation, we have analyzed the behaviour of the test under the null for $AR(2)$ processes (X_t) and (Y_t) with parameters $(a_1, a_2) = (0, 0.8)$ and standard normally distributed white noise, i.e. $X_t = a_2 X_{t-2} + e_t$ and $Y_t = a_2 Y_{t-2} + \epsilon_t$. To study the size, we have compared

(X_t) with an AR(2) process with parameters $(a_1, a_2) = (0, 0.2)$ [cf. Figure 2]. Both are referred to as model (IIa) and (IIb) respectively. This has been done for the the same sample sizes and weights that have been used for the first setting in Figure 1. We have chosen such AR(2) models, because the largest difference of f_X and f_Y is around the frequency $\frac{\pi}{2}$ which is more preferrably weighted for the sequence of weights d_j.

The corresponding simulation results can be found in Figure 2 and Table 2. Again we can make similar observations as for the first models: The approximation under the null works very well. Although the tests have more power than in the first situation they again do not seem to be consistent. Note that the power only increases slightly from $(n_1, n_2) = (200, 300)$ to $(n_1, n_2) = (800, 1200)$ in the upper right panel of Figure 2.

5. Conclusions

In Section 3 we have investigated the distributional limits of $T_n^{(3)}$ and $T_n^{(4)}$ under H_0. We will now comment on the statistical worth of corresponding tests. As already remarked above the limit of $T_n^{(3)}$ under H_0 depends on unknown values of the spectra. Hence the above results cannot be applied to derive an asymptotic exact test for hypotheses with boundary H_0.
In comparison to that the limit of $T_n^{(4)}$ has the advantage to be distribution-free under H_0. Denote the corresponding test by $\varphi_n := \mathbf{1}_{(c(\alpha),\infty)}(T_n^{(4)})$, where the critical value $c(\alpha)$ is given by the $(1-\alpha)$-quantile of the distribution of $S^{(4)}$. Remark that Theorem 3.2 shows that φ_n is an asymptotical level α test. However, as we have seen in the last section the test performs poorly under the alternative. We can even show that its inherited structure causes this inconsistency, i.e. we do not have $E(\varphi_n) \to 1$ under the alternative. Moreover, it is even worse since it can be proven in a similar way to Theorem 3.2 that

$$T_n^{(4)} \xrightarrow{\mathcal{D}} \sum_{j=1}^{\infty} c_j \log^2\left(\frac{f_X(\omega_j)}{f_Y(\omega_j)} \frac{E_j}{\hat{E}_j}\right) \stackrel{\mathcal{D}}{=} \sum_{j=1}^{\infty} c_j \left(\log\left(\frac{f_X(\omega_j)}{f_Y(\omega_j)}\right) + Z_j\right)^2 \tag{5.1}$$

holds even if H_0 is not fulfilled. Here $E_j, \hat{E}_j$ are i.i.d. standard exponential distributed r.v.s.. As can be seen from this general convergence result, the above test statistic cannot distingiush between H_0 and general alternatives. In fact, the test has asymptotic power

$$P\left(\sum_{j=1}^{\infty} c_j \left(\log\left(\frac{f_X(\omega_j)}{f_Y(\omega_j)}\right) + Z_j\right)^2 > c_\alpha\right) \tag{5.2}$$

under H_1 which is in general not equal to 1. A solution may be to use consistent estimators like integrated periodograms, see e.g. Dahlhaus [7], or smoothed periodograms, see Eichler (2008) as well as Dette and Paparoditis (2009) for the case of equal sample sizes. However, since the consistency of these estimators comes along with the usage of Fourier frequencies, again the problem of an adequate treatment of different sample sizes arises.
To sum up, the at first sight natural approach to use periodogram-based distances for testing hypotheses about different spectras has not turned out to be very promising. As we have seen this is mainly caused by the inconsistency of the periodograms. Since it is not clear how to translate the tests in Eichler (2008) and Dette and Paparoditis (2009) to the setting with unequal sample sizes, a satisfactory test for testing the null hypothesis of equal spectras is still unknown in this case.

References

[1] Billingsley, P. (1968). Convergence of probability measures. John Wiley & Sons, New York.

[2] Brillinger, D.R. (1981). Time series. Data analysis and theory. Holden-Day Series in Time Series Analysis, San Francisco.
[3] Brockwell, P.J. and Davis, R.A. (1991). Time series: theory and methods. 2nd ed. Springer-Verlag, Berlin.
[4] Caiado, J., Crato, N. and Peña, D. (2006). A periodogram-based metric for time series classification. Comput. Stat. Data Anal. 50, 2668-2684.
[5] Caiado, J., Crato, N. and Peña, D. (2009). Comparison of times series with unequal length in the frequency domain. Commun. Stat., Simulation Comput. 38, 527-540.
[6] Coates, D.S. and Diggle, P.J. (1986). Tests for comparing two estimated spectral densities. J. Time Ser. Anal. 7, 7-20.
[7] Dahlhaus, R. (1985). Asymptotic normality of spectral estimates. J. Mult. Anal. 16, 412-431.
[8] Dette, H. and Paparoditis, E. (2009). Bootstrapping frequency domain tests in multivariate time series with an application to comparing spectral densities. J. R. Stat. Soc. B, 71, 831-857.
[9] Diggle, P.J. and Fisher, N.I. (1991). Nonparametric comparison of cumulative periodograms. J. R. Stat. Soc. C 40, 423-434.
[10] Eichler, M. (2008). Testing nonparametric and semiparametric hypotheses in vector stationary processes. J. Multivariate Anal. 99, 968-1009.
[11] Maharaj, E.A. (2002). Comparison of non-stationary time series in the frequency domain. Comput. Stat. Data Anal. 40, 131-141.
[12] Pötscher, B.M. and Reschenhofer, E. (1988). Discriminating between two spectral densities in case of replicated observations. J. Time Ser. Anal. 9, 221-224.
[13] Shorack, G.R. and Wellner J.A. (1986). Empirical processes with applications to statistics. John Wiley & Sons, New York.

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